REAL ANALYSIS PRELIM 2017

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may use any standard theorem from your real analysis course, either identifying it by name or stating it in full.

In the following, $\mathbb{R}$ *denotes the real numbers, and* $\mathbb{N}$ *stands for the set of positive integers.*

Problem 1. For $k = 0, 1, 2, \ldots$, prove the following two identities involving Lebesgue integrals.

(a) The gamma function identity:

$$\int_0^\infty y^k e^{-y} dy = k!\,.$$

(b) The incomplete gamma function identity for any $t > 0$:

$$\frac{1}{k!} \int_t^\infty y^k e^{-y} dy = e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!}\,.$$

Problem 2. Suppose H is a Hilbert space and $y_n \in H$ are such that the sequence $\langle x, y_n \rangle$ converges, as $n \to \infty$, for every $x \in H$. Prove that there exists $y \in H$ such that $\lim_{n\to\infty} \langle x, y_n \rangle = \langle x, y \rangle$ for every $x \in H$.

Problem 3. Suppose X, Y are Hausdorff spaces and X is compact. Let $f : X \to Y$ be a map and let $G(f) = \{(x, f(x)),\ x \in X\}$ be the graph of f in $X \times Y$. Equip $X \times Y$ with the product topology and $G(f)$ with the subspace topology. Prove that f is continuous if and only if $G(f)$ is compact.

Problem 4. Consider the closed unit interval $[0, 1]$, and the Banach space $C[0, 1]$ of continuous real-valued functions on $[0, 1]$ with the supremum norm.

(a) For any $f \in C[0, 1]$ and any $\epsilon > 0$, show that the closed ball

$$D(f, \epsilon) = \{g \in C[0, 1] :\ \|g - f\| \le \epsilon\}$$

is not compact.

(b) Prove that $C[0, 1]$ is not σ-compact; that is, prove that $C[0, 1]$ cannot be expressed as a countable union of compact subsets.

Problem 5. A function f is said to be convex on an interval $[a, b]$ if for all $x, y \in [a, b]$ and $0 \le \lambda \le 1$ we get

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y).$$

For a convex function $f : [a, b] \to \mathbb{R}$, prove the following.

(a) If $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}.$$

(b) The left-hand and right-hand derivatives of f exist at every point of (a, b), and f must be continuous on (a, b).

Problem 6. Consider $\mathbb{R}$, equipped with the Lebesgue measure. Suppose that $f \in L^\infty(\mathbb{R})$ is such that, for every $g \in L^1(\mathbb{R})$ and every $a \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} g(x)\,(f(x + a) - f(x))\,dx = 0.$$

Prove that there exists a constant c such that $f(x) = c$ for almost all $x \in \mathbb{R}$.

Problem 7. A metric space is said to be separable if it has a countable dense subset. Let ℓ^∞ be the space of bounded sequences of real numbers. Turn it to a metric space with the sup-norm metric. That is, given two bounded sequences $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$, define

$$d(\alpha, \beta) = \sup_{i\in\mathbb{N}} |\alpha_i - \beta_i|\,.$$

Prove that this metric space is not separable.

Problem 8. Let a be an arbitrary constant and and $b \ge 0$. Let f be a nonnegative Borel measurable function that is bounded on bounded intervals and satisfies

$$0 \le f(t) \le a + b\int_0^t f(s)\,ds, \quad \text{for all } t \ge 0.$$

Use recursion to prove that $f(t) \le ae^{bt}$ for all $t \ge 0$.