# REAL ANALYSIS PRELIMINARY EXAM

March, 2019

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it. Please start each solution on a new page and submit your solutions in order.

If any questions or instructions are not clear, ask the proctor.

You may use any standard theorem from your real analysis course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

The notation $\mathbb{R}$ stands for the real numbers. For $p \geq 1$, $L^p(\mathbb{R}^n)$ denotes the $L^p$-space on $\mathbb{R}^n$ with respect to Lebesgue measure. Given an integer $k \geq 1$, we say a function defined on an open subset of $\mathbb{R}^n$ is $C^k$ if it has continuous derivatives up to and including the $k$th order. The notation $f \in C([0,1])$ means $f$ is a continuous function on $[0,1]$.

**Problem 1.** Let $g \in L^2(\mathbb{R})$, and set $f(x) = \int_0^x g(t)dt$, $x \in \mathbb{R}$.

(a) Show by an example that $f$ need not be differentiable at 0.

(b) Must $f$ have any points of differentiability? Explain.

(c) Let $\varphi(x) = f(x)^2$. Show that $\varphi$ is differentiable at 0 and find $\varphi'(0)$.

**Problem 2.** Let $\{f_n : n \geq 1\}$ be a sequence of continuously differentiable functions on $[0,1]$, and assume that

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \text{ for all } 0 < x \leq 1 \text{ and all } n \geq 1,$$

and that

$$\int_0^1 f_n(x)\, dx = 0 \text{ for all } n \geq 1.$$

Prove that this sequence has a subsequence that converges uniformly on [0, 1].

**Problem 3.** Consider $\mathbb{R}$, equipped with the Lebesgue measure. Suppose that $f \in L^\infty(\mathbb{R})$ is such that, for every $g \in L^1(\mathbb{R})$ and every $a \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} g(x)[f(x+a) - f(x)]\, dx = 0.$$

Prove that there exists a constant $c$ such that $f(x) = c$ for almost all $x \in \mathbb{R}$

**Problem 4.** Suppose that $H$ is a separable real Hilbert space with an orthonormal basis $\{e_k : k \geq 1\}$ and with inner product denoted by $\langle \cdot, \cdot \rangle$. Let $\{y_k : k \geq 1\} \subset H$. Prove that the following two statements are equivalent.

(a) $\lim_{k\to\infty}\langle x, y_k\rangle = 0$ for every $x \in H$;
(b) $\sup_{k\geq 1}\|y_k\| < \infty$ and $\lim_{k\to\infty}\langle e_n, y_k\rangle = 0$ for every $n \geq 1$.

**Problem 5.** Let $\mathcal{B}$ be a real Banach space. Denote by $\|\cdot\|$ the norm on $\mathcal{B}$. Suppose that $\mathcal{B}$ satisfies the best approximation property, that is:

Given a closed subspace $\mathcal{M}$ of $\mathcal{B}$ and given $x \in \mathcal{B}$ there exists $y_x \in \mathcal{M}$ such that $\|x - y_x\| = \inf_{y\in\mathcal{M}}\|y - x\| = \mathrm{dist}\,(x, \mathcal{M})$.

Prove that for every bounded linear function $f \in \mathcal{B}^*$ there exists $z \in \mathcal{B}$ such that $f(z) = \|f\|\|z\|$.

**Problem 6.** Let $K$ be a continuous function on $[0,1]\times[0,1]$. For $f \in L^2([0,1])$ and $x \in [0,1]$, define

$$Tf(x) = \int_0^1 K(x,y)f(y)dy.$$

(a) Show that $Tf$ is continuous on $[0,1]$ and $\|Tf\|_\infty \leq \|K\|_\infty\|f\|_2$ for all $x \in [0,1]$. Here $\|\cdot\|_\infty$ denotes the $L^\infty$-norm.
(b) Show that if $\{f_n : n \geq 1\}$ is a bounded sequence in $L^2([0,1])$, the sequence $\{Tf_n : n \geq 1\}$ has a uniformly convergent subsequence.
(c) Assume that $T$ is one-to-one. Show that $T$ does not map $L^2([0,1])$ onto $C([0,1])$.

**Problem 7.** Suppose $f \in L^1([0,1])$. Show that if $\int_0^1 f(x)(\sin x)^n dx = 0$ for every $n = 1, 2, 3, \ldots$, then $f = 0$ a.e. on $[0,1]$.

**Problem 8.** The Fourier cosine transform of a function $f \in L^1(\mathbb{R})$ is the function on $\mathbb{R}$ defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) f(t)\, dt$$

Prove the "Riemann-Lebesgue lemma": if $f \in L^1(\mathbb{R})$ then $\hat{f}$ is continuous and vanishes at infinity. You can assume without proof that functions of class $C^1$ that vanish outside a bounded interval are dense in $L^1(\mathbb{R})$.