REAL ANALYSIS PRELIMINARY EXAM

September, 2018

Do as many of the eight problems as you can. If any questions or instructions are not clear, ask the proctor. Four completely correct solutions will be regarded as a clear pass. Keep in mind that complete solutions are better than partial solutions. It also helps in partial solutions to clearly indicate where the gaps are. Always carefully justify your answers.

You may use any standard theorem from your real analysis course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

The notation $\mathbb{R}$ stands for the real numbers. For $p \geq 1$, $L^p(\mathbb{R}^n)$ denotes the L^p-space on $\mathbb{R}^n$ with respect to Lebesgue measure. For an integer $k \geq 1$, we say a function defined on an open subset of $\mathbb{R}^n$ is C^k if it has continuous derivatives up to and including the kth order. The notation $f \in C([0,1])$ means f is a continuous function on $[0,1]$.

Problem 1. Let μ be a regular Borel measure on $\mathbb{R}^n$ such that $\mu(B) < \infty$ for any ball $B \subset \mathbb{R}^n$. Let E be a Borel set with $\mu(E) > 0$ such that

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty \quad \text{for all } x \in E.$$

Show that there exist a closed set $E_0 \subset E$ with $0 < \mu(E_0) < \infty$ and constants $r_0 > 0$ and $M \geq 1$ such that for all $x \in E_0$ and $0 < r < r_0$

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq M.$$

Problem 2. Let m denote Lebesgue measure in $\mathbb{R}^n$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant L, that is,

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x,\, y \in \mathbb{R}^n,$$

where $|x - y|$ denotes the usual Euclidean distance. Show that for every Borel set E,

$$m(f(E)) \leq L^n m(E).$$

Problem 3. Let F be an increasing right-continuous real-valued function on $[0,1]$ with $F(0) = 0$. Prove that F can be written as

$$F = F_A + F_S + F_J,$$

where the functions F_A, F_S, and F_J are all increasing, $F_A(0) = F_S(0) = F_J(0) = 0$, and

(i) F_A is absolutely continuous,
(ii) F_S is continuous and singular, that is, $F_S'(x) = 0$ for a.e. $x \in [0,1]$, and

(iii) F_J is a right-continuous "jump function," meaning that there is an empty, finite, or countable subset $S \subset (0,1]$ and positive constants $\{c_s\}_{s\in S}$ with $\sum_{s\in S} c_s < \infty$ for which

$$F_J(x) \;=\; \sum_{\{s\in S\,:\,s\le x\}} c_s \quad \text{for all } x \in [0,1],$$

where an empty sum is 0.

Moreover, show that these functions F_A, F_S, and F_J are uniquely determined by F.

Problem 4. Let $(E, \mathcal{F}, \mu)$ be a probability measure space, that is, μ is a positive measure on $(E, \mathcal{F})$ and $\mu(E) = 1$. Suppose that $\{X_n : n \ge 1\}$ is a sequence of measurable functions on $(E, \mathcal{F}, \mu)$ that is uniformly integrable; that is, for every $\varepsilon > 0$, there is some $M > 0$ so that

$$\int_{\{|X_n|>M\}} |X_n| d\mu < \varepsilon \quad \text{for every } n \ge 1.$$

Show that $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_E \sup_{1\le k\le n} |X_k| d\mu = 0.$

Problem 5. Suppose that X and Y are Hausdorff topological spaces and that X is compact.

(a) Show that a mapping $f : X \to Y$ is continuous if and only if its graph

$$\Gamma = \{x, f(x)) : x \in X\}$$

is compact in $X \times Y$ in the product topology.

(b) Show that the above assertion cannot be weakened to Γ being closed in $X \times Y$. That is, find a compact Hausdorff space X, a Hausdorff space Y, and a mapping $f : X \to Y$ for which Γ is closed and f is not continuous.

Problem 6. Let $1 < p < \infty$, and $\{f_n\}$ be a sequence of functions in $C^1((0,1)) \cap C([0,1])$, satisfying

$$\int_0^1 |f_n(x)|^p \, dx + \int_0^1 |f_n'(x)|^p \, dx \le 1 \quad \text{for every } n \ge 1.$$

(a) Show that there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in L^p([0,1]) \cap C([0,1])$ such that

$$f_{n_k} \to f \text{ uniformly in } [0,1], \text{ and } f_{n_k} \to f \text{ in } L^p([0,1]), \text{ as } k \to \infty.$$

(b) Show that there exist a sub-subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ and a function $h \in L^p([0,1])$ such that for every $\varphi \in C_c((0,1))$
(that is, φ is continuous with compact support in $(0,1)$),

$$\lim_{j\to\infty} \int_0^1 f'_{n_{k_j}}(x)\varphi(x)\,dx = \int_0^1 h(x)\varphi(x)\,dx.$$

(c) Show that for every $\varphi \in C_c^1((0,1))$
(that is, φ is C^1 with compact support in $(0,1)$),

$$\int_0^1 h(x)\varphi(x)\,dx = -\int_0^1 f(x)\varphi'(x)\,dx.$$

Problem 7. Suppose that $f \in L^1(\mathbb{R}^4)$.

(a) Show that the integral

$$\Phi(x) = \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^2}\,dy$$

is absolutely convergent for a.e. $x \in \mathbb{R}^4$ and that the resulting function Φ is integrable in any bounded measurable subset of $\mathbb{R}^4$.

(b) Show that if $f \in C_c^2(\mathbb{R}^4)$ (that is, f is C^2 and compactly supported), then $\Phi \in C^2(\mathbb{R}^4)$ and

$$\Delta\Phi(x) = \sum_{i=1}^4 \frac{\partial^2\Phi}{\partial x_i^2} = -2\sigma_3 f(x) \quad \text{for every } x \in \mathbb{R}^4,$$

where σ_3 is the surface area of the unit sphere in $\mathbb{R}^4$.

Problem 8. Let A be a closed linear operator on a real Hilbert space H; that is, A is a linear operator on H and its graph $\Gamma(A) = \{(x, Ax) : x \in H)\}$ is closed in the product Hilbert space $H \times H$.

(a) Define the adjoint operator A^* as usual: if $y \in H$ and there exists a $z \in H$ such that for all $x \in H$, $\langle Ax, y\rangle = \langle x, z\rangle$, then y is in the domain of A^* and we define $A^*y = z$. Show that A^* is a bounded linear operator defined on all of H.

(b) Show that for every $a, b \in H$, the system of equations

$$\begin{cases} x + A^*y = a \\ Ax - y = b \end{cases}$$

has a unique solution $x, y \in H$.