Math 583A, Spring 2017

MWF 10:30- 11:20

Sklyanin Algebras

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Course description. The Sklyanin algebras first arose in mathematical physics in the development of the quantum inverse scattering method. They have been intensively studied since then by mathematicians and by mathematical physicists. One of the main open questions in Sklyanin's original paper was to determine their irreducible representations. This has now been done (more or less), the key tool being non-commutative projective algebraic geometry. We will gradually approach this central question and use the Sklyanin algebras as a vehicle for motivating non-commutative projective algebraic geometry. The idea is that suitable non-commutative graded algebras should be thought of as homogeneous coordinate rings of "non-commutative projective schemes".

En route, we will introduce an eclectic mix of material including parts of the cohomological machinery underpinning modern algebraic geometry, the representation theory of ${}_2$, of quantum ${}_2$, elliptic curves, theta functions, and non-commutative algebra.

In addition to the three 50-minute lectures each week there is a 90-minute discussion session each week focused on examples, homework problems, and student questions. We append a list of possible topics, but we would like to build some flexibility into the course, so that audience requests might alter the plan. There is a possibility that students might give presentations on some of the topics.

1. Some non-commutative rings. The Weyl algebra=differential operators on the affine line. Enveloping algebras generally. $U({}_2)$ and its quotients embedded in the Weyl algebra. Recollection of some ${}_2$ representation theory, including Verma modules, and some aspects of geometric representation theory (realization of representations as sections of bundles on the flag variety).
2. The q-deformed enveloping algebra $U_q({}_2)$ (a "quantum group"), treated in parallel with the classical ${}_2$ as above.
3. Graded rings and modules. Bergman's Diamond Lemma, and Hilbert series.
4. Quotient categories and their relation to open/closed subschemes of varieties.

5. Graded algebras and graded modules in projective algebraic geometry. The functors $M \mapsto$ and $\mapsto \bigoplus H^0(X, (n))$.
6. Serre's Theorem that the category of quasi-coherent sheaves on a projective variety is equivalent to a certain quotient category of the category of graded modules over a homogeneous coordinate ring.
7. Zhang twists and equivalences of graded module categories. Some non-commutative homogeneous coordinate rings of the projective line P^1. E.g., the q-deformed polynomial ring $C_q[x, y]$ in which $yx = qxy$, $q \in -\{0, 1\}$.
8. Homological aspects: Koszul, Artin-Schelter, and Gorenstein, properties. Few proofs.
9. Non-commutative projective geometry, non-commutative analogues of the projective plane ^{2}, the quadric $^1 \times ^1$, and ^{3}. Point modules and line modules.
10. Homogenization of $U(_2)$ and $U_q(_2)$, and recovery of some of their representation theory from non-commutative projective geometry, specifically non-commutative analogues of ^{3}.
11. Yang-Baxter equation from mathematical physics as motivation for Sklyanin algebras.
12. Sklyanin algebras as introduced in E. K. Sklyanin's papers.
13. Parallels between the representation theory of Sklyanin algebras, $_2$, and quantum $_2$.
14. Theta functions in one variable as needed for $/\Lambda \to^2$ and $/\Lambda \to^3$ and actions of Sklyanin algebras on spaces of theta functions as difference operators or infinite order differential operators.

Some relevant papers.

- D. Keeler, The rings of noncommutative projective geometry, arXiv:0205005v2
- D. Rogalski, An introduction to noncommutative projective algebraic geometry, arxiv:1403.3065v1.
- E. K. Sklyanin. Some algebraic structures connected with the Yang-Baxter equation. *Funktsional. Anal. i Prilozhen.*, **16(4)** (1982) 27-34.
- E. K. Sklyanin. Some algebraic structures connected with the Yang-Baxter equation. Representations of a quantum algebra. *Funktsional. Anal. i Prilozhen.*, **17(4)** (1983) 34-48.
- S.P. Smith, The 4-dimensional Sklyanin algebras, *K-Theory,* **8** (1994) 65–80.

- J. T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc.*, **38** (2001) 171-216. arXiv:math/9910082

Prerequisites. See instructor.