INSUFFICIENCY OF THE BRAUER-MANIN OBSTRUCTION FOR RATIONAL POINTS ON ENRIQUES SURFACES

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ABSTRACT. In [VAV11], Várilly-Alvarado and the last author constructed an Enriques surface X over $\mathbb{Q}$ with an étale-Brauer obstruction to the Hasse principle and no *algebraic* Brauer-Manin obstruction. In this paper, we show that the nontrivial Brauer class of $X_{\overline{\mathbb{Q}}}$ does not descend to $\mathbb{Q}$. Together with the results of [VAV11], this proves that the Brauer-Manin obstruction is insufficient to explain all failures of the Hasse principle on Enriques surfaces.

The methods of this paper build on the ideas in [CV15, CV15b, IOOV]: we study geometrically unramified Brauer classes on X via pullback of ramified Brauer classes on a rational surface. Notably, we develop techniques which work over fields which are not necessarily separably closed, in particular, over number fields.

1. INTRODUCTION

Given a smooth, projective, geometrically integral variety X over a global field k, one may ask whether X has a k-rational point, that is, whether $X(k) \neq \emptyset$. Since k embeds into each of its completions, a necessary condition for $X(k) \neq \emptyset$ is that $X(\mathbb{A}_k) \neq \emptyset$. However, this condition is often not sufficient; varieties X with $X(\mathbb{A}_k) \neq \emptyset$ and $X(k) = \emptyset$ exist, and these are said to fail the Hasse principle.

In 1970, Manin [Man71] significantly advanced the study of failures of the Hasse principle by use of the Brauer group and class field theory. More precisely, he defined a subset $X(\mathbb{A}_k)^{\mathrm{Br}}$ of $X(\mathbb{A}_k)$, now known as the Brauer-Manin set, with the property that

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

Thus, we may think of an empty Brauer-Manin set as an obstruction to the existence of rational points. At the time of its introduction, this Brauer-Manin obstruction explained all known failures of the Hasse principle.

In 1999, Skorobogatov [Sko99] defined a refinement of the Brauer-Manin set, the étale-Brauer set $X(\mathbb{A}_k)^{\mathrm{ét,Br}}$, which still contains $X(k)$. He proved that this new obstruction can be stronger than the Brauer-Manin obstruction, by constructing a bielliptic surface $X/\mathbb{Q}$ such that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{ét,Br}} = \emptyset$ and $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$.

Bielliptic surfaces have a number of geometric properties in common with Enriques surfaces: both have Kodaira dimension 0 and nontrivial étale covers. This raises the natural

question of whether the étale-Brauer obstruction is stronger than the Brauer-Manin obstruction *for Enriques surfaces*. Harari and Skorobogatov took up this question in 2005; they constructed an Enriques surface $X/\mathbb{Q}$ whose étale-Brauer set was strictly smaller than the Brauer-Manin set [HS05], thereby showing that the Brauer-Manin obstruction is insufficient to explain all failures of weak approximation[1] on Enriques surfaces. Their surface, however, had a $\mathbb{Q}$-rational point, so it did not fail the Hasse principle.

The main result of this paper is the analogue of Harari and Skorobogatov's result for the Hasse principle. Precisely, we prove:

Theorem 1.1. *The Brauer-Manin obstruction is insufficient to explain all failures of the Hasse principle on Enriques surfaces.*

This theorem builds on work of Várilly-Alvarado and the last author. To explain the connection, we must first provide more information about the Brauer group. For any variety X/k, we have the following filtration of the Brauer group:

$$\mathrm{Br}_0\, X := \mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, X) \subset \mathrm{Br}_1\, X := \ker(\mathrm{Br}\, X \to \mathrm{Br}\, X_{k^{\mathrm{sep}}}) \subset \mathrm{Br}\, X = \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbb{G}_m).$$

Elements in $\mathrm{Br}_0\, X$ are said to be constant, elements in $\mathrm{Br}_1\, X$ are said to be algebraic, and the remaining elements are said to be transcendental. The Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$ depends only on the quotient $\mathrm{Br}\, X/\mathrm{Br}_0\, X$ (this follows from the fundamental exact sequence of class field theory, see [Sko01, §5.2] for more details). As transcendental Brauer elements have historically been difficult to study, one sometimes instead considers the (possibly larger) algebraic Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}_1}$, defined in terms of the subquotient $\mathrm{Br}_1\, X/\mathrm{Br}_0\, X$.

We now recall the main result of [VAV11].

Theorem ([VAV11, Thm. 1.1])**.** *There exists an Enriques surface $X/\mathbb{Q}$ such that*
$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{\acute{e}t,Br}} = \emptyset \quad \text{and} \quad X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset.$$

The proof of [VAV11, Thm. 1.1] is constructive. Precisely, for any $\mathbf{a} = (a, b, c) \in \mathbb{Z}^3$ with
$$abc(5a + 5b + c)(20a + 5b + 2c)(4a^2 + b^2)(c^2 - 100ab)(c^2 + 5bc + 10ac + 25ab) \neq 0, \quad (1.1)$$
the authors consider $Y_{\mathbf{a}} \subset \mathbb{P}^5$, the smooth degree-8 K3 surface given by
$$v_0 v_1 + 5v_2^2 = w_0^2,$$
$$(v_0 + v_1)(v_0 + 2v_1) = w_0^2 - 5w_1^2,$$
$$av_0^2 + bv_1^2 + cv_2^2 = w_2^2.$$
The involution $\sigma \colon \mathbb{P}^5 \to \mathbb{P}^5, (v_0 : v_1 : v_2 : w_0 : w_1 : w_2) \mapsto (-v_0 : -v_1 : -v_2 : w_0 : w_1 : w_2)$ has no fixed points on $Y_{\mathbf{a}}$ so the quotient $X_{\mathbf{a}} := Y_{\mathbf{a}}/\sigma$ is an Enriques surface.

Theorem ([VAV11, Theorem 1.2])**.** *Let $\mathbf{a} = (a, b, c) \in \mathbb{Z}^3_{>0}$ satisfy the following conditions:*
  (1) *for all prime numbers $p \mid (5a + 5b + c)$, 5 is not a square modulo p,*
  (2) *for all prime numbers $p \mid (20a + 5b + 2c)$, 10 is not a square modulo p,*
  (3) *the quadratic form $av_0^2 + bv_1^2 + cv_2^2 + w_2^2$ is anisotropic over $\mathbb{Q}_3$,*
  (4) *the integer $-bc$ is not a square modulo 5,*
  (5) *the triplet (a, b, c) is congruent to $(5, 6, 6)$ modulo 7,*

---

[1]A smooth projective variety X satisfies weak approximation if $X(k)$ is dense in $X(\mathbb{A}_k)$ in the adelic topology.

(6) *the triplet (a, b, c) is congruent to $(1, 1, 2)$ modulo 11,*
(7) $Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset,$
(8) *the triplet (a, b, c) is* Galois general *(meaning that a certain number field defined in terms of a, b, c is as large as possible).*

Then

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{ét,Br}} = \emptyset \quad and \quad X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{Br}_1} \neq \emptyset.$$

Várilly-Alvarado and the last author deduce [VAV11, Thm. 1.1] from [VAV11, Thm. 1.2] by showing that the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8). Henceforth, when we refer to "conditions" by number, we mean the conditions given in the theorem above.

In [VAV11], the authors have left open the question of a transcendental obstruction to the Hasse principle for the surfaces $X_{\mathbf{a}}$, due to the "difficulty [...] in finding an explicit representative for [the nontrivial] Brauer class of $[\overline{X}_{\mathbf{a}}]$." Recent work of Creutz and the last author [CV15, CV15b], and Ingalls, Obus, Ozman, and the last author [IOOV] makes this problem more tractable. Building on techniques from [CV15, CV15b, IOOV], we prove:

Theorem 1.2. *If $\mathbf{a} = (a, b, c) \in \mathbb{Z}_{>0}^3$ satisfies conditions (5), (6), and (8) then* $\text{Br} \, X_{\mathbf{a}} = \text{Br}_1 \, X_{\mathbf{a}}$. *In particular, if $\mathbf{a}$ satisfies conditions (1)–(8), then*

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{ét,Br}} = \emptyset \quad and \quad X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{Br}} \neq \emptyset.$$

Strategy and outline. Theorem 1.1 and the second statement of Theorem 1.2 both follow immediately from the first statement of Theorem 1.2 and [VAV11], since the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8) [VAV11, Lemma 6.1 and Proof of Theorem 1.1]. Thus, we reduce to proving the first statement of Theorem 1.2.

For any variety X over a field k, the quotient $\text{Br} \, X / \text{Br}_1 \, X$ injects into $(\text{Br} \, X_{k^{\text{sep}}})^{\text{Gal}(k^{\text{sep}}/k)}$. In Skorobogatov's pioneering paper, his construction $X/\mathbb{Q}$ had the additional property that $(\text{Br} \, X_{\overline{\mathbb{Q}}})^{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = 0$, so $\text{Br} \, X = \text{Br}_1 \, X$. Unfortunately, this strategy cannot be applied to an Enriques surface X, as $\text{Br} \, X_{k^{\text{sep}}} \cong \mathbb{Z}/2\mathbb{Z}$ [HS05, p. 3223] and hence the unique nontrivial element is always fixed by the Galois action.

Instead, we will find a Galois extension $K_1/\mathbb{Q}$ and an open set $U' \subset X_{\mathbf{a}}$ such that

(1) the **geometrically unramified subgroup** $\text{Br}^{\text{g.unr.}} \, U'_{K_1} \subset \text{Br} \, U'_{K_1}$ (i.e., the subgroup of elements in $\text{Br} \, U'_{K_1}$ which are contained in $\text{Br} \, \overline{X}_{\mathbf{a}} \subset \text{Br} \, \overline{U'}$ upon base change to $\overline{\mathbb{Q}}$) surjects onto $\text{Br} \, \overline{X}_{\mathbf{a}}$, and
(2) $(\text{Br}^{\text{g.unr.}} \, U'_{K_1} / \text{Br} \, K_1)^{\text{Gal}(K_1/\mathbb{Q})}$ is contained in $\text{Br}_1 \, U'_{K_1} / \text{Br} \, K_1$.

The key step is proving (2) *without* necessarily having central simple $\mathbf{k}(U'_{K_1})$-algebra representatives for all of the elements of $\text{Br}^{\text{g.unr.}} \, U'_{K_1} / \text{Br} \, K_1$. Our approach follows the philosophy laid out in [CV15, CV15b, IOOV]: we study geometrically unramified Brauer classes on U'_{K_1} via pullback of ramified Brauer classes on simpler surface S', of which U' is a double cover. However, in contrast to the work of [CV15, CV15b, IOOV], we carry this out over a field that is not necessarily separably closed. In particular, our methods can be carried out over a number field. As we expect this approach to be of independent interest, we build up some general results in §2 which can be applied to a double cover of a rational ruled surface, assuming mild conditions on the branch locus.

Remark 1.3. For convenience, we carry out the above strategy on the K3 surface $Y_{\mathbf{a}}$ instead of on the Enriques surface $X_{\mathbf{a}}$. We then descend the results to $X_{\mathbf{a}}$.

Starting in §3, we restrict our attention to the specific varieties $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$. After recalling relevant results from [VAV11], we construct double cover maps $\pi \colon Y_{\mathbf{a}} \to S$ and $\tilde{\pi} \colon X_{\mathbf{a}} \to \widetilde{S}$, where S and $\widetilde{S}$ are ruled surfaces, and we study the geometry of these morphisms. These maps allow us to apply the results of [CV15] to construct, in §4, an explicit central simple $\mathbf{k}(\overline{X}_{\mathbf{a}})$-algebra representative $\mathcal{A}$ of the nontrivial Brauer class of $\overline{X}_{\mathbf{a}}$. This representative $\mathcal{A}$ will necessarily be defined over a number field K_1, be unramified over an open set U'_{K_1}, and be geometrically unramified. Furthermore, the number field K_1 and the open set U' can be explicitly computed from the representative $\mathcal{A}$.

Section 5 uses the results from §2 to study the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K_1)$ on $\mathrm{Br}^{\mathrm{g.unr.}} U'_{K_1}/\mathrm{Br}\, K_1$ and hence prove Theorem 1.2. Namely, by repeated application of the commutative diagram in Theorem 2.2, we demonstrate that no σ-invariant transcendental Brauer class can exist for $Y_{\mathbf{a}}$. Indeed, if such a class existed, the explicit central simple algebra from §4 would relate it to a function $\tilde{\ell}$ fixed by the Galois action. However, direct computation (given in the appendix) shows that $\tilde{\ell}$ must be moved by some Galois action, providing the required contradiction.

General notation. Throughout, k will be a field with characteristic not equal to 2, with fixed separable closure $\bar{k}$. For any k-scheme X and field extension k'/k, we write $X_{k'}$ for the base change $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$ and $\overline{X}$ for the base change $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$. If X is integral, we write $\mathbf{k}(X)$ for the function field of X. We also denote the absolute Galois group by $G_k = \mathrm{Gal}(\bar{k}/k)$. For any k-variety W, we use the term splitting field (of W) to mean the smallest Galois extension over which every geometrically irreducible component of W is defined.

The Picard group of X is $\mathrm{Pic}\, X := \mathrm{Div}\, X/\mathrm{Princ}\, X$, where $\mathrm{Div}\, X$ is the group of Weil divisors on X and $\mathrm{Princ}\, X$ is the group of principal divisors on X; when X is projective, $\mathrm{Pic}\, X$ is representable by a scheme, called the Picard scheme [Gro62, Cor. 6.6, p. 232-17]. If X is projective, let $\mathrm{Pic}^0 X$ denote the subgroup of $\mathrm{Pic}\, X$ that maps to the connected component of the identity in the Picard scheme of X; then the Néron-Severi group of X is $\mathrm{NS}\, X := \mathrm{Pic}\, X/\mathrm{Pic}^0 X$. For a divisor $D \in \mathrm{Div}\, X$, we write $[D]$ for its equivalence class in $\mathrm{Pic}\, X$. When X is a curve, the Jacobian of X satisfies $\mathrm{Jac}\, X = \mathrm{Pic}^0 X$.

For a k-scheme Y, we write $\mathrm{Br}\, Y$ for the étale cohomology group $\mathrm{Br}\, Y := \mathrm{H}^2_{\mathrm{ét}}(Y, \mathbb{G}_m)$. If Y is projective, we additionally consider the geometrically unramified subgroup $\mathrm{Br}^{\mathrm{g.unr.}}\, \mathbf{k}(Y) \subset \mathrm{Br}\, \mathbf{k}(Y)$ consisting of those Brauer classes which are contained in $\mathrm{Br}\, \overline{Y}$ upon base change to $\bar{k}$. For an open subscheme $U \subset Y$, we have $\mathrm{Br}^{\mathrm{g.unr.}}\, U := \mathrm{Br}\, U \cap \mathrm{Br}^{\mathrm{g.unr.}}\, \mathbf{k}(Y)$. If A is an étale k-algebra, then we write $\mathrm{Br}\, A$ for $\mathrm{Br}(\mathrm{Spec}\, A)$. Given invertible elements a and b in such an A, we define the quaternion algebra $(a, b) := A[i, j]/\langle i^2 = a, j^2 = b, ij = -ji \rangle$. We will identify the algebra (a, b) with its class in $\mathrm{Br}\, A$.

Now assume that Y is smooth. Then the following sequence is exact:

$$0 \to \mathrm{Br}\, Y[2] \to \mathrm{Br}\, \mathbf{k}(Y)[2] \xrightarrow{\oplus_y \partial_y} \bigoplus_y \mathrm{H}^1(\mathbf{k}(y), \mathbb{Z}/2\mathbb{Z}), \tag{1.2}$$

where the sum is taken over the set of all codimension-1 points y on Y [Gro68, Thm. 6.1]. As $\mathrm{Br}\, \mathbf{k}(Y)[2]$ is generated by quaternion algebras, we will only describe the residue map ∂_y on quaternion algebras: for any $a, b \in \mathbf{k}(Y)^\times$, we have

$$\partial_y((a, b)) = (-1)^{v_y(a)v_y(b)} a^{v_y(b)} b^{-v_y(a)} \in \mathbf{k}(y)^\times/\mathbf{k}(y)^{\times 2} \cong \mathrm{H}^1(\mathbf{k}(y), \mathbb{Z}/2\mathbb{Z}),$$

where v_y denotes the valuation corresponding to y ; as $\mathbf{k}(y)^\times/\mathbf{k}(y)^{\times 2} \cong \mathrm{H}^1\left(\mathbf{k}(y), \mathbb{Z}/2\mathbb{Z}\right)$, we move freely between additive and multiplicative notation when computing residues, depending on the context.

2. BRAUER CLASSES ON DOUBLE COVERS ARISING VIA PULLBACK

Let $\pi^0 \colon Y^0 \to S^0$ be a double cover of a smooth projective rational geometrically ruled surface $\varpi \colon S^0 \to \mathbb{P}^1_t$ defined over k and let $B^0 \subset S^0$ denote the branch locus of π^0. (Throughout, $\mathbb{P}^1_t$ is shorthand for $\mathbb{P}^1_{[t_0:t_1]}$, with $t := t_0/t_1$.) We assume that B^0 is reduced, geometrically irreducible, and has at worst ADE singularities. The canonical resolution [BHPVdV84, Thm. 7.2] $\nu \colon Y \to Y^0$ of $\pi^0 \colon Y^0 \to S^0$ has a 2-to-1 k-morphism $\pi \colon Y \to S$ to a smooth rational *generically* ruled surface S; the branch curve $B \subset S$ of π is a smooth proper model of B^0. In summary, we have the following diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \pi\ } & S \ \supset B \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu_S} \\
Y^0 & \xrightarrow{\ \pi^0\ } & S^0 \supset B^0
\end{array}
$$

Since B^0 is geometrically irreducible, $\mathrm{Pic}^0 Y$ is trivial by [CV15, Cor. 6.3] and so we may conflate $\mathrm{Pic}\, Y$ and $\mathrm{NS}\, Y$.

The generic fiber of $\varpi \circ \pi^0$ is a double cover $C \to \mathbb{P}^1_{k(t)} \to \mathrm{Spec}\, k(t)$. Since $k(t)$ is infinite, by changing coordinates if necessary, we may assume that the double cover is unramified above $\infty \in S^0_{k(t)}$. Then C has a model

$$
y^2 = c'h(x),
$$

for some $c' \in k(t)$ and $h \in k(t)[x]$ square-free, monic, and with $\deg(h) = 2g(C) + 2$, where $g(C)$ denotes the genus of C. Note that $\mathbf{k}(B) = \mathbf{k}(B^0) \cong k(t)[\theta]/(h(\theta))$; we write α for the image of θ in $\mathbf{k}(B)$.

As S^0 is rational and geometrically ruled, $\mathrm{Pic}\, S^0 \cong \mathbb{Z}^2$ and is generated by a fibre S^0_∞, and a section $\mathfrak{S}$, which we may take to be the closure of $x = \infty$. Since $\nu \colon S \to S^0$ is a birational map, $\mathrm{Pic}\, \overline{S}$ is generated by the strict transforms of $\mathfrak{S}$ and S^0_∞ and the curves $E_1, \ldots, E_n$ which are contracted by the map $S \to S^0$. We will often abuse notation and conflate $\mathfrak{S}$ and S^0_∞ with their strict transforms. In any case, by $\mathfrak{S}$, B, E_i, or S^0_∞, we always mean the actual divisors and *not* the divisor classes in the Picard group.

Let

$$
\mathcal{E} = \left\{\mathfrak{S}, S^0_\infty, E_1, \ldots, E_n\right\}
$$

denote the aforementioned set of $n+2$ generators and define

$$V := S \setminus \left(B \cup \bigcup_{E \in \mathcal{E}} E \right) \subset S.$$

Possibly after replacing k with a finite extension, we may assume that all elements of $\mathcal{E}$ are defined over k and, in particular, that $\operatorname{Pic} S = \operatorname{Pic} \overline{S}$. Since ν_S is defined over k, we additionally have that S is k-rational and so $\operatorname{Br} S = \operatorname{Br} S^0 = \operatorname{Br} k$.

For any $\ell \in \mathbf{k}(B)^\times$, we define

$$\mathcal{A}_\ell := \operatorname{Cor}_{\mathbf{k}(B)(x)/k(t,x)} \left((\ell, x - \alpha) \right) \in \operatorname{Br} \mathbf{k}(S).$$

We will be particularly concerned with $\mathcal{A}_\ell$ when ℓ is contained in the subgroup

$$\mathbf{k}(B)_{\mathcal{E}}^\times := \left\{ \ell \in \mathbf{k}(B)^\times : \operatorname{div}(\ell) \in \operatorname{im}(\mathbb{Z}^{\mathcal{E}} \to \operatorname{Div}(S) \to \operatorname{Div}(B) \otimes \mathbb{Z}/2\mathbb{Z}) \right\}$$
$$= \left\{ \ell \in \mathbf{k}(B)^\times : \nu_* \operatorname{div}(\ell) \in \langle \mathfrak{S}, S_\infty^0 \rangle \subset \operatorname{Div}(B^0) \otimes \mathbb{Z}/2\mathbb{Z} \right\}.$$

By [CV15, Proof of Thm. 5.2], this subgroup is exactly the set of functions ℓ such that $\pi^* \mathcal{A}_\ell$ is geometrically unramified. Note that $\mathbf{k}(B)_{\mathcal{E}}^\times$ contains $k^\times \mathbf{k}(B)^{\times 2}$.

Let

$$U := Y \setminus \left(\bigcup_{E \in \mathcal{E}} \pi^{-1}(E) \right) \subset Y.$$

The goal of this section is to prove the following two theorems.

Theorem 2.1. *Let k' be any Galois extension of k. Then we have the following exact sequence of $\operatorname{Gal}(k'/k)$-modules:*

$$0 \to \frac{\operatorname{Pic} Y_{k'}}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y_{k'}} \xrightarrow{j} \frac{\mathbf{k}(B_{k'})_{\mathcal{E}}^\times}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} \xrightarrow{\beta} \left(\frac{\operatorname{Br}^{\text{g. unr.}} U_{k'}}{\operatorname{Br} k'} \right) [2], \qquad (2.1)$$

where j is as in §2.3 and β is as in §2.2. Furthermore, if k' is separably closed, then the last map surjects onto $\operatorname{Br} Y[2]$.

Theorem 2.2. *We retain the notation from Theorem 2.1. If $\operatorname{Br} k' \to \operatorname{Br} \mathbf{k}(S_{k'})$ is injective and $\operatorname{Pic} \overline{U}[2] = 0$, then there is a commutative diagram of $\operatorname{Gal}(k'/k)$-modules with exact rows and columns:*

The structure of the section is as follows. In §2.1, we prove some preliminary results about the residues of $\mathcal{A}_\ell$; these are used in section §2.2 to define the map β. Next, in §2.3, we define j and prove that it is injective. In §2.4, we characterize the elements of $\operatorname{Br} V$ that pull back to constant algebras under π^*. In §2.5, we combine the results from the earlier sections to prove Theorem 2.1, and, finally, in §2.6, we prove Theorem 2.2.

2.1. **Residues of $\mathcal{A}_\ell$.** In order to define the homomorphism β, we will need to know certain properties about the residues of $\mathcal{A}_\ell$ at various divisors of S^0. We first compute residues associated to horizontal divisors.

Lemma 2.3. *Let $\ell \in \mathbf{k}(B)^\times$, and let F be an irreducible horizontal curve in S^0, i.e., a curve that maps dominantly onto $\mathbb{P}_t^1$.*

    (1) *If $F \neq B, \mathfrak{S}$, then $\partial_F(\mathcal{A}_\ell) = 1 \in \mathbf{k}(F)^\times/\mathbf{k}(F)^{\times 2}$.*
    (2) *$\partial_B(\mathcal{A}_\ell) = [\ell] \in \mathbf{k}(B)^\times/\mathbf{k}(B)^{\times 2}$.*

Proof. The arguments in this proof follow those in [CV15b, Proofs of Thm. 1.1 and Prop. 2.3]; as the situation is not identical, we restate the arguments here for the reader's convenience.

Let v be the valuation on $\mathbf{k}(S^0)$ associated to F. By [CV15b, Lemma 2.1], we have

$$\partial_F(\mathcal{A}_\ell) = \prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)}\left((-1)^{w(\ell)w(x-\alpha)}\ell^{w(x-\alpha)}(x-\alpha)^{-w(\ell)}\right), \qquad (2.2)$$

where w runs over all valuations on $\mathbf{k}(B \times_{\mathbb{P}_t^1} S^0)$ extending v. As F is a horizontal divisor, $v|_{k(t)}$ is trivial and hence $w|_{\mathbf{k}(B)}$ is trivial for all $w|v$. Therefore, (2.2) simplifies to $\prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)}(\ell^{w(x-\alpha)})$.

By definition of α, $\mathrm{Norm}_{\mathbf{k}(B)(x)/k(t)(x)}(x-\alpha) = h(x)$. Thus, $w(x-\alpha) = 0$ for all $w|v$ if $v(h(x)) = 0$, or equivalently, if $F \neq B, \mathfrak{S}$. This completes the proof of (1).

Now assume that $F = B$. We know that $h(x)$ factors as $(x-\alpha)h_1(x)$ over $\mathbf{k}(B)(x)$, where $h_1 \in k(t)[x]$ is possibly reducible. Hence, $x - \alpha$ determines a valuation $w_{x-\alpha}$ on $\mathbf{k}(B)(x)$ lying over v; similarly, the other irreducible factors of h_1 also determine valuations lying over v. Notice that since $h(x)$ is separable (as B is reduced), we have that $h_1(\alpha) \neq 0$, and hence that $w(x - \alpha) = 0$ for any valuation w over v corresponding to the irreducible factors of $h_1(x)$. Thus, (2.2) simplifies to

$$\prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)}(\ell^{w(x-\alpha)}) = \mathrm{Norm}_{\mathbf{k}(w_{x-\alpha})/\mathbf{k}(v)}(\ell) = \ell,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now we compute the residues associated to vertical divisors.

Lemma 2.4. *Let $\ell \in \mathbf{k}(B)_{\mathcal{E}}^\times$, $t_0 \in \mathbb{A}_t^1 \subset \mathbb{P}_t^1$ be a closed point, and $F = S_{t_0}^0$. Then,*

$$\partial_F(\mathcal{A}_\ell) \in \mathrm{im}\left(\frac{\mathbf{k}(t_0)^\times}{\mathbf{k}(t_0)^{\times 2}} \to \frac{\mathbf{k}(F)^\times}{\mathbf{k}(F)^{\times 2}}\right).$$

Remark 2.5. If k is separably closed, then $\mathbf{k}(t_0)^{\times 2} = \mathbf{k}(t_0)^\times$ and the result follows from [CV15b, Prop. 3.1].

Proof. It suffices to show that $\partial_F(\mathcal{A}_\ell) \in \mathbf{k}(F)^{\times 2}\mathbf{k}(t_0)^\times$. We repeat [CV15, Proof of Prop. 3.1] while keeping track of scalars to accommodate the fact that k is not necessarily separably closed.

By [CV15b, Lemma 2.1], we have

$$\partial_F(\mathcal{A}_\ell) = \prod_{\substack{F' \subset S^0 \times_{\mathbb{P}^1_t} B \\ F' \mapsto F \text{ dominantly}}} \mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)}(x-\alpha)^{-w'(\ell)}), \qquad (2.3)$$

where F' is an irreducible curve and w' denotes the valuation associated to F'. The surface $S^0 \times_{\mathbb{P}^1_t} B$ is a geometrically ruled surface over B, so the irreducible curves F' are in one-to-one correspondence with points $b' \in B$ mapping to t_0. Furthermore, $\mathbf{k}(F') = \mathbf{k}(b')(x)$ and $\mathbf{k}(F) = \mathbf{k}(t_0)(x)$, so $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$ is induced from $\mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}$. Thus, we may rewrite (2.3) as

$$\partial_F(\mathcal{A}_\ell) = \prod_{b' \in B, b' \mapsto t_0} \mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)}(x-\alpha)^{-w'(\ell)}). \qquad (2.4)$$

By [CV15, Lemma 3.3], there exists an open set $W \subset \mathbb{A}^1$ containing t_0 and constants $d \in \mathbf{k}(t)^\times$, $e \in \mathbf{k}(t)$ such that

$$S^0_W \to \mathbb{P}^1_k \times W, \quad s \mapsto (dx(s) + e, \varpi(s))$$

is an isomorphism. In particular, $dx + e$ is a horizontal function on S^0_W. Consider the following equality:

$$\mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}\left((dx + e - (d\alpha + e), \ell)\right) = \mathcal{A}_\ell + \mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}((d, \ell)) = \mathcal{A}_\ell + (d, \mathrm{Norm}(\ell)).$$

Since $(d, \mathrm{Norm}(\ell)) \in \varpi^* \mathrm{Br}\, k(t)$, we have

$$\partial_F(\mathcal{A}_\ell) \in \partial_F\left(\mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}\left((dx + e - (d\alpha + e), \ell)\right)\right) \mathbf{k}(t_0)^\times.$$

Thus, we may assume that x is a horizontal function, in particular, that x has no zeros or poles along F, and that it restricts to a non-constant function along F. It is then immediate that $w'(x - \alpha) \le 0$, and that the inequality is strict if and only if $w'(\alpha) < 0$, which in turn happens if and only if b' lies over $B^0_{t_0} \cap \mathfrak{S}$.

We first consider the factor of (2.4) that corresponds to points that do not lie over $B^0_{t_0} \cap \mathfrak{S}$. If b' does not lie over $B^0_{t_0} \cap \mathfrak{S}$, then (as stated above) $w'(x - \alpha) = 0$, where w' denotes the valuation associated to b'. Therefore, the corresponding factor of (2.4) simplifies to

$$\prod_{b' \in B \setminus \nu^{-1}(B^0 \cap \mathfrak{S}), b' \mapsto t_0} \mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}\left((x - \alpha(b'))^{-w'(\ell)}\right).$$

By definition, $\ell \in \mathbf{k}(B)^\times_{\mathcal{E}}$ implies that for all $b'' \in B^0 \setminus (B^0 \cap \mathfrak{S})$, $\sum_{b' \in B, b' \mapsto b''} w'(\ell) \equiv 0 \bmod 2$. Since $\alpha(b')$ depends only on the image of b' in B^0, this shows that the above factor is contained in $\mathbf{k}(F)^{\times 2}$.

Now consider the case that b' lies over $B^0_{t_0} \cap \mathfrak{S}$. We claim that, since $w'(x) = 0$,

$$\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}\left((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)}(x-\alpha)^{-w'(\ell)}\right) \qquad (2.5)$$

reduces to a constant in $\mathbf{k}(F)$. Indeed, if $w'(\ell) = 0$, then we obtain $\ell^{w'(x-\alpha)}$, which reduces (after taking $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$) to an element of $\mathbf{k}(t_0)^\times$. If $w'(\ell) \ne 0$, let $\pi_{F'}$ be a uniformizer for F'. Since $w'(x) = 0 > w'(\alpha)$, we have

$$\left(\frac{\ell}{\pi_{F'}^{w'(\ell)}}\right)^{w'(x-\alpha)} \left(\frac{x-\alpha}{\pi_{F'}^{w'(x-\alpha)}}\right)^{-w(\ell)} = \left(\frac{\ell}{\pi_{F'}^{w'(\ell)}}\right)^{w'(x-\alpha)} \left(\frac{-\alpha}{\pi_{F'}^{w'(x-\alpha)}}\right)^{-w(\ell)} \quad \bmod \pi_{F'}$$

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and so (2.5) reduces to (again, after taking $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$) an element in $\mathbf{k}(t_0)^\times$. Thus, every factor of (2.4) corresponding to points b' lying over $B^0_{t_0} \cap \mathfrak{S}$ is contained in $\mathbf{k}(t_0)^\times$, and every other factor is an element of $\mathbf{k}(F)^{\times 2}$. This completes the proof. $\qquad\square$

2.2. The morphism β.

Proposition 2.6. *Let $\ell \in \mathbf{k}(B)^\times$. There exists an element $\mathcal{A}' = \mathcal{A}'(\ell) \in \mathrm{Br}\, k(t)$, unique modulo $\mathrm{Br}\, k$, such that*

$$\mathcal{A}_\ell + \varpi^* \mathcal{A}' \in \mathrm{Br}\, V.$$

This induces a well-defined homomorphism

$$\beta \colon \frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{k^\times \mathbf{k}(B)^{\times 2}} \to \frac{\mathrm{Br}^{\mathrm{g.unr.}}\, U}{\mathrm{Br}\, k}[2], \quad \ell \mapsto \pi^* \left(\mathcal{A}_\ell + \varpi^* \mathcal{A}' \right),$$

which is surjective if k is separably closed.

Proof. Recall that $V = S \setminus \left(B \cup \bigcup_{E \in \mathcal{E}} E \right) \subset S$. Therefore, as a subgroup of $\mathrm{Br}\, \mathbf{k}(S) = \mathrm{Br}\, \mathbf{k}(S^0)$, $\mathrm{Br}\, V$ is equal to $\mathrm{Br}\, \left(S^0 \setminus (\mathfrak{S} \cup S^0_\infty \cup B) \right)$, since the Brauer group of a surface is unchanged under removal of a codimension 2 closed subscheme [Gro68, Thm. 6.1]. Thus, to prove the first statement, it suffices to show that there exists an element $\mathcal{A}' \in \mathrm{Br}\, k(t)$, unique up to constant algebras, such that $\partial_F(\mathcal{A}_\ell) = \partial_F(\varpi^* \mathcal{A}')$ for all irreducible curves $F \subset S^0$ with $F \neq \mathfrak{S}, S^0_\infty, B$.

If F is any horizontal curve, i.e., F maps dominantly to $\mathbb{P}^1_t$, then $\partial_F(\varpi^* \mathcal{A}') = 1$ for all $\mathcal{A}' \in \mathrm{Br}\, k(t)$. If we further assume that $F \neq \mathfrak{S}, B$, then Lemma 2.3 gives $\partial_F(\mathcal{A}_\ell) = 1$. Thus, for all $\mathcal{A}' \in \mathrm{Br}\, k(t)$, we have $\partial_F(\mathcal{A}_\ell) = \partial_F(\varpi^* \mathcal{A}')$ for all horizontal curves $F \neq \mathfrak{S}, B$.

Now we turn our attention to the vertical curves. Recall Faddeev's exact sequence [GS06, Cor. 6.4.6]:

$$0 \to \mathrm{Br}\, k \to \mathrm{Br}\, k(t) \xrightarrow{\oplus \partial_{t_0}} \bigoplus_{t_0 \in \mathbb{P}^1_t} \mathrm{H}^1(G_{\mathbf{k}(t_0)}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sum_{t_0} \mathrm{Cor}_{\mathbf{k}(t_0)/k}} \mathrm{H}^1(G_k, \mathbb{Q}/\mathbb{Z}) \to 0. \qquad (2.6)$$

Since the residue field at $t_0 = \infty$ is equal to k, this sequence implies that for any element $(r_{t_0}) \in \oplus_{t_0 \in \mathbb{A}^1} \mathbf{k}(t_0)^\times / \mathbf{k}(t_0)^{\times 2}$, there exists a Brauer class in $\mathcal{A}' \in \mathrm{Br}\, k(t)$, unique modulo elements of $\mathrm{Br}\, k$, such that $\partial_{t_0}(\mathcal{A}') = r_{t_0}$ for all closed points $t_0 \in \mathbb{A}^1$. By Lemma 2.4, for all $t_0 \in \mathbb{A}^1$, we have $\partial_F(\mathcal{A}_\ell) \in \mathrm{im}\, (\mathbf{k}(t_0)^\times / \mathbf{k}(t_0)^{\times 2} \to \mathbf{k}(F)^\times / \mathbf{k}(F)^{\times 2})$, where $F = S^0_{t_0}$. Hence, there exists a $\mathcal{A}' \in \mathrm{Br}\, k(t)$, unique modulo $\mathrm{Br}\, k$, such that $\partial_F(\varpi^* \mathcal{A}') = \partial_F(\mathcal{A}_\ell)$ for all $F \neq \mathfrak{S}, B, S^0_\infty$, as desired.

It remains to prove the second statement. The first statement immediately implies the existence of a well-defined homomorphism

$$\frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{\mathbf{k}(B)^{\times 2}} \to \frac{\mathrm{Br}\, \pi^{-1}(V)}{\mathrm{Br}\, k}[2], \quad \ell \mapsto \pi^* \left(\mathcal{A}_\ell + \varpi^* \mathcal{A}' \right).$$

In order to complete the proof, we must prove that

    (1) $\pi^* (\mathcal{A}_d + \varpi^* \mathcal{A}') \in \mathrm{Br}\, k$ if $d \in k^\times$,
    (2) the image lands in $\mathrm{Br}^{\mathrm{g.unr.}}\, U / \mathrm{Br}\, k$, and
    (3) the image is equal to $\mathrm{Br}\, Y[2]$ if k is separably closed.

We begin with (1). Let $d \in k^\times$. Then

$$\mathcal{A}_d = \mathrm{Cor}_{\mathbf{k}(B)(x)/k(t,x)}(d, x - \alpha) = (d, \mathrm{Norm}_{\mathbf{k}(B)(x)/k(t,x)}(x - \alpha)) = (d, h(x)) = (d, c'h(x)) + (d, c').$$

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Since $\sqrt{c'h(x)}$ generates $\mathbf{k}(Y^0)/\mathbf{k}(S^0)$, $\operatorname{div}(c'h(x)) = B + 2Z$ for some divisor Z on S^0. Thus, $(d, c'h(x))$ is unramified away from B; in particular, $(d, c'h(x)) \in \operatorname{Br} V$. Since $\mathcal{A}'$ is the unique element in $\operatorname{Br} k(t)/\operatorname{Br} k$ such that $\mathcal{A}_d + \mathcal{A}' \in \operatorname{Br} V$, then $\mathcal{A}' = (d, c') + \mathcal{B}$ for some $\mathcal{B} \in \operatorname{Br} k$. Hence,

$$\pi^* \left(\mathcal{A}_d + \varpi^* \mathcal{A}' \right) = \pi^* \left((d, c'h(x)) + (d, c') + (d, c') + \pi^* \varpi^* \mathcal{B} \right) = \pi^* (d, c'h(x)) + \pi^* \varpi^* \mathcal{B}.$$

Furthermore, since $c'h(x)$ is a square in $\mathbf{k}(Y^0)$, then $\pi^* \left(\mathcal{A}_d + \varpi^* \mathcal{A}' \right) = \pi^* \varpi^* \mathcal{B} \in \operatorname{Br} k$, as desired.

Now we turn to (2) and (3). Since B is the branch locus of π and π is 2-to-1, any 2-torsion Brauer class in $\operatorname{im} \left(\pi^* \colon \operatorname{Br} \mathbf{k}(S) \to \operatorname{Br} \mathbf{k}(Y) \right)$ is unramified at $\pi^{-1}(B)_{\mathrm{red}}$. Thus, the image is contained in $\operatorname{Br} U / \operatorname{Br} k$. To prove that it is contained in $\operatorname{Br}^{\mathrm{g.unr.}} U$, we must show that $\pi^* \left(\mathcal{A}_\ell + \varpi^* \mathcal{A}' \right)_{\overline{k}}$ is contained in $\operatorname{Br} \overline{Y}$. By Tsen's theorem, $\pi^* \left(\mathcal{A}_\ell + \varpi^* \mathcal{A}' \right)_{\overline{k}} = (\pi^* \mathcal{A}_\ell)_{\overline{k}}$. This element is contained in $\operatorname{Br} \overline{Y}$ by [CV15, Thm. I], which yields (2). In fact, [CV15, Thm. I] shows that $\operatorname{Br} \overline{Y}[2]$ is generated by $\pi^* \mathcal{A}_\ell$ where ℓ runs over the elements in $\mathbf{k}(B_{\overline{k}})_{\mathcal{E}}$, which proves (3). $\qquad\square$

2.3. **The morphism** j. In this section, we define the map j and prove that it is injective. The map j will be induced by the following homomorphism:

$$j' \colon \operatorname{Div}(Y \setminus \pi^{-1}(B)) \to \mathbf{k}(B)^\times / k^\times$$
$$D \mapsto \ell|_B$$

where $\ell \in \mathbf{k}(S)^\times$ is such that $\operatorname{div}_S(\ell) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS_\infty^0$. (Recall that $E_1, \ldots, E_n, \mathfrak{S}$, and S_∞^0 form an integral basis for $\operatorname{Pic} S = \operatorname{Pic} \overline{S}$.)

Lemma 2.7. *The homomorphism j' induces a well-defined injective homomorphism*

$$j \colon \frac{\operatorname{Pic} Y}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y} \to \frac{\mathbf{k}(B)_{\mathcal{E}}^\times}{k^\times \mathbf{k}(B)^{\times 2}}.$$

Proof. For any divisor $D \in \operatorname{Div} Y \setminus \pi^{-1}(B)$, the projection formula [Liu02, p.399] yields

$$2\pi_* (D \cap \pi^{-1}(B)_{\mathrm{red}}) = \pi_* (D \cap 2\pi^{-1}(B)_{\mathrm{red}}) = \pi_* (D \cap \pi^*(B)) = (\pi_* D) \cap B.$$

Thus, for any divisor $D \in \operatorname{Div} Y \setminus \pi^{-1}(B)$, we have that $[D \cap \pi^{-1}(B)_{\mathrm{red}}] \in \left(\frac{\operatorname{Pic} B}{\operatorname{im} \operatorname{Pic} S \to \operatorname{Pic} B} \right)[2]$. By the same argument as in proof of [IOOV, Lemma 4.8], this induces a well-defined injective homomorphism

$$\frac{\operatorname{Pic} Y}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y} \to \left(\frac{\operatorname{Pic} B}{\operatorname{im} \operatorname{Pic} S \to \operatorname{Pic} B} \right)[2], \quad [D] \mapsto [D \cap \pi^{-1}(B)_{\mathrm{red}}]. \qquad (2.7)$$

One can also check that there is a well-defined injective homomorphism

$$\left(\frac{\operatorname{Pic} B}{\operatorname{im} \operatorname{Pic} S \to \operatorname{Pic} B} \right)[2] \to \frac{\mathbf{k}(B)_{\mathcal{E}}^\times}{k^\times \mathbf{k}(B)^{\times 2}} \qquad (2.8)$$

that sends a divisor D which represents a class in $\left(\frac{\operatorname{Pic} B}{\operatorname{im} \operatorname{Pic} S \to \operatorname{Pic} B} \right)[2]$ to a function ℓ such that $\operatorname{div}(\ell) = 2D + \sum_{C \in \operatorname{Pic} S} n_C C \cap B$. Since j is the composition of (2.7) and (2.8), this completes the proof that j is well-defined and injective. $\qquad\square$

2.4. Brauer classes on V that become constant under π^*.

Proposition 2.8. *If $\mathcal{A} \in \operatorname{Br} V$ is such that $\pi^*\mathcal{A} \in \operatorname{Br} k \subset \operatorname{Br} \mathbf{k}(Y)$, then there exists a divisor $D \in \operatorname{Div} Y$ such that $j([D]) = \partial_B(\mathcal{A})$ in $\mathbf{k}(B)^\times / k^\times \mathbf{k}(B)^{\times 2}$.*

Proof. Recall that $\mathbf{k}(Y_k) = \mathbf{k}(S_k)(\sqrt{c'h(x)})$. Thus, if $\pi^*\mathcal{A} \in \operatorname{Br} k$, then

$$\mathcal{A} = (c'h(x), G) + \mathcal{B} \tag{2.9}$$

for some $G \in \mathbf{k}(S_k)^\times$ and some $\mathcal{B} \in \operatorname{Br} k$. Since B is the branch locus of π, $v_B(c'h(x))$ must be odd. Therefore, without loss of generality, we may assume that B is not contained in the support of G; write

$$\operatorname{div}(G) = \sum_i n_i C_i + d(\mathfrak{S}) + e(S_\infty^0) + m_1 E_1 + \cdots + m_n E_n,$$

where C_i are k-irreducible curves of S distinct from $\mathfrak{S}, S_\infty^0$, and $E_1, \ldots, E_n$.

Now we consider the residue of $\mathcal{A}$ at C_i. By (2.9), the residue of $\mathcal{A}$ at C_i is $[c'h(x)] \in \mathbf{k}(C_i)^\times / \mathbf{k}(C_i)^{\times 2}$. On the other hand, $\mathcal{A} \in \operatorname{Br} V$, so the residue is trivial at C_i. Together, these statements imply that $\pi^{-1}(C_i)$ consists of 2 irreducible components C_i' and C_i''. As this is true for all C_i, we have that $\operatorname{div}(G) = \pi_*(\sum_i n_i C_i') + m(\mathfrak{S}) + m_0(S_\infty^0) + m_1 E_1 + \cdots + m_n E_n$, and so $j'(\sum n_i C_i') = G|_B$ modulo $k^\times$. Since the residue of $\mathcal{A}$ at B is equal to $G|_B$, this completes the proof. $\qquad\qquad\square$

2.5. Proof of Theorem 2.1.

We note that much of this proof is very similar to proofs in [IOOV, Lemmas 4.4 and 4.8].

We will first prove the sequence is exact, and then show that the maps are compatible with the Galois action. Since all assumed properties of k are preserved under field extension, we may, for the moment, assume that $k = k'$. Then Lemma 2.7 yields an injective homomorphism

$$j \colon \frac{\operatorname{Pic} Y_{k'}}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y_{k'}} \longrightarrow \frac{\mathbf{k}(B_{k'})_{\mathcal{E}}^\times}{k'^\times \mathbf{k}(B_{k'})^{\times 2}},$$

and Proposition 2.6 yields a homomorphism

$$\beta \colon \frac{\mathbf{k}(B_{k'})_{\mathcal{E}}^\times}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} \longrightarrow \left(\frac{\operatorname{Br}^{\text{g. unr.}} U_{k'}}{\operatorname{Br} k'} \right)[2],$$

which is surjective if k' is separably closed. We now show that $\operatorname{im}(j) = \ker(\beta)$.

Let $\ell \in \mathbf{k}(B_{k'})_{\mathcal{E}}^\times$ be such that $\beta(\ell) \in \operatorname{Br} k'$. Recall that β factors through $\operatorname{Br} V / \operatorname{Br} k$ by the map

$$\ell \mapsto \underbrace{\mathcal{A} := \mathcal{A}_\ell + \varpi^*\mathcal{A}'}_{\in \operatorname{Br} V / \operatorname{Br} k} \mapsto \pi^*\mathcal{A},$$

where $\mathcal{A}' \in \operatorname{Br} k'(t)$ is as in Proposition 2.6. By assumption, $\pi^*\mathcal{A} \in \operatorname{Br} k$, thus, by Proposition 2.8, there is some $D \in \operatorname{Div} Y_k$ such that $j([D]) = \partial_B(\mathcal{A}) = \partial_B(\mathcal{A}_\ell)\partial_B(\varpi^*\mathcal{A}') \bmod k^\times$. However, $\partial_B(\varpi^*\mathcal{A}') = 1$ since B is a horizontal divisor, and $\partial_B(\mathcal{A}_\ell) = [\ell]$ by Lemma 2.3. Hence, $\ell \in \operatorname{im}(j)$, and so $\operatorname{im}(j) \supset \ker(\beta)$.

For the opposite inclusion, it suffices to prove that $\beta(j([D])) \in \operatorname{Br} k'$ for any prime divisor $D \in \operatorname{Div}(Y_{k'} \setminus \pi^{-1}(B))$. Let $\ell = j'(D)$; recall that ℓ is the restriction to B of a function

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$\ell_S \in \mathbf{k}(S_{k'})$ such that $\mathrm{div}(\ell_S) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS_\infty^0$. As above, let $\mathcal{A} := \mathcal{A}_\ell + \varpi^* \mathcal{A}'$. We claim that

$$\mathcal{A} = (c'h(x), \ell_S) + \mathcal{B} \quad \in \mathrm{Br}\,\mathbf{k}(S_{k'}) = \mathrm{Br}\,\mathbf{k}(S_{k'}^0)$$

for some $\mathcal{B} \in \mathrm{Br}\,k'$. Since $c'h(x) \in \mathbf{k}(Y_{k'})^{\times 2}$, this equality implies that $\pi^*(\mathcal{A}) = \pi^*\mathcal{B} \in \mathrm{Br}\,k'$. To prove the claim, we will compare residues of $\mathcal{A}$ and $(c'h(x), \ell_S)$ on S^0. Repeated application of Faddeev's exact sequence [GS06, Cor. 6.4.6] shows that $\mathrm{Br}\,\mathbb{A}_{k'}^n$ is trivial; in particular, the Brauer group of $S^0 \setminus (S_\infty^0 \cup \mathfrak{S})$, which is isomorphic to $\mathbb{A}^2$, consists only of constant algebras. Hence, if $\mathcal{A} - (c'h(x), \ell_S)$ is unramified everywhere on $S^0 \setminus (\mathfrak{S} \cup S_\infty^0)$, then it must be constant. By Lemma 2.3(2) and the assumption that $\mathcal{A} \in \mathrm{Br}\,V$, it suffices to show that $\partial_B((c'h(x), \ell_S)) = [\ell]$ and that $(c'h(x), \ell_S)$ is unramified along all other curves irreducible curves contained in V.

Let R be a prime divisor of S^0 different from $S_\infty^0, \mathfrak{S}$, and B. Since B is the branch locus of π, we may assume that $v_R(c'h(x)) = 0$. Hence, $\partial_R((c'h(x), \ell_S)) = (c'h(x))^{v_R(\ell_S)}$. Now, we know that

$$\mathrm{div}_S(\ell_S) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS_\infty^0.$$

Thus, if $R \neq \pi(D)$ then $v_R(\ell_S) = 0$ and hence $\partial_R((c'h(x), \ell_S))$ is trivial. It remains to consider the case $R = \pi(D)$. Note that $\pi_*(D)$ is equal to $\pi(D)$ if π maps D isomorphically to its image, and is equal to $2\pi(D)$ otherwise. In the latter case, we must have that $v_{\pi(D)}(\ell_S)$ is even, meaning that $\partial_{\pi(D)}((c'h(x), \ell_S))$ is trivial (up to squares). On the other hand, if $\pi_*(D) = \pi(D)$, then $c'h(x)$ must be a square in $\mathbf{k}(\pi(D))$, and hence $\partial_{\pi(D)}((c'h(x), \ell_S))$ is again trivial (up to squares).

Finally, $\partial_B((c'h(x), \ell_S)) = [\ell_S|_B] = [\ell]$. Indeed, since we know that

$$\mathrm{div}(\ell_S) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS_\infty^0$$

for some $D \in \mathrm{Div}(Y \setminus \pi^{-1}(B))$, we see that B is not in the support of $\mathrm{div}(\ell_S)$. Hence $v_B(\ell_S) = 0$. Moreover, since $v_B(c'h(x))$ is odd, the usual residue formula allows us to deduce that $\partial_B((c'h(x), \ell_S)) = [\ell]$ modulo squares. Hence, $\mathcal{A} = (c'h(x), \ell_S) + \mathcal{B}$ and the sequence is exact, as desired.

Now we consider the Galois action. That j respects the Galois action is clear from the definition. To see that β is a homomorphism of Galois modules, we note that every geometrically irreducible curve outside of V is irreducible over k. Thus, the residue maps ∂_F for F outside of V are defined over k, which shows that β is a homomorphism of Galois modules. This completes the proof. $\qquad\square$

2.6. **Proof of Theorem 2.2.** Applying Theorem 2.1 to $k' = \overline{k}$ and $k = k'$ and taking the subgroups of Galois invariant elements gives an exact sequence

$$0 \longrightarrow \left(\frac{\mathrm{Pic}\,\overline{Y}}{\pi^*\,\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}} \right)^{G_{k'}} \xrightarrow{j_{\overline{k}}} \left(\frac{\mathbf{k}(\overline{B})_{\mathcal{E}}^{\times}}{\mathbf{k}(\overline{B})^{\times 2}} \right)^{G_{k'}} \xrightarrow{\beta_{\overline{k}}} (\mathrm{Br}\,\overline{Y})^{G_{k'}}. \qquad (2.10)$$

Recall that $j_{\overline{k}}$ factors through $\mathbf{k}(\overline{S})/\mathbf{k}(\overline{S})^{\times 2}$, so the middle term may be replaced with $\left(\mathbf{k}(\overline{B})_{\mathcal{E}}/\mathbf{k}(\overline{B})^{\times 2} \right)^{G_{k'}} \cap \mathrm{im}(\mathbf{k}(\overline{S})/\mathbf{k}(\overline{S})^{\times 2})^{G_{k'}}$.

To determine $(\mathbf{k}(\overline{S})/\mathbf{k}(\overline{S})^{\times 2})^{G_{k'}}$, we consider the exact sequence

$$0 \longrightarrow \overline{k}^{\times} \longrightarrow \mathbf{k}(\overline{S})^{\times} \longrightarrow \mathbf{k}(\overline{S})^{\times}/\overline{k}^{\times} \longrightarrow 0.$$

After taking the cohomological long exact sequence, applying inflation-restriction and Hilbert's Theorem 90 (twice), and applying the assumption that $\operatorname{Br} k' \longrightarrow \operatorname{Br} \mathbf{k}(S_{k'})$ is injective, we obtain

$$\mathrm{H}^0\left(G_{k'}, \mathbf{k}(\overline{S})^\times/\overline{k}^\times\right) = \mathbf{k}(S_{k'})^\times/k'^\times, \quad \text{and} \quad \mathrm{H}^1\left(G_{k'}, \mathbf{k}(\overline{S})^\times/\overline{k}^\times\right) = 0.$$

Then the cohomological long exact sequence associated to

$$0 \longrightarrow \mathbf{k}(\overline{S})^\times/\overline{k}^\times \xrightarrow{\times 2} \mathbf{k}(\overline{S})^\times/\overline{k}^\times \longrightarrow \mathbf{k}(\overline{S})^\times/\mathbf{k}(\overline{S})^{\times 2} \longrightarrow 0$$

yields $\left(\mathbf{k}(\overline{S})^\times/\mathbf{k}(\overline{S})^{\times 2}\right)^{G_{k'}} = \mathbf{k}(S_{k'})/(k'^\times \mathbf{k}(S_{k'})^{\times 2})$. Thus, we may replace (2.10) with

$$0 \longrightarrow \left(\frac{\operatorname{Pic}\overline{Y}}{\pi^*\operatorname{Pic} S + 2\operatorname{Pic}\overline{Y}}\right)^{G_{k'}} \xrightarrow{j_{\overline{k}}} \frac{\mathbf{k}(B_{k'})_{\mathcal{E}}}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} \xrightarrow{\beta_{\overline{k}}} (\operatorname{Br}\overline{Y})^{G_{k'}}.$$

Note that by Proposition 2.6 $\beta_{\overline{k}}|_{\mathbf{k}(B_{k'})_{\mathcal{E}}}$ factors through $\operatorname{Br}^{\mathrm{g.unr.}} U_{k'}/\operatorname{Br} k'$. Hence, we obtain the following commutative diagram:

It remains to prove that the rows and columns are exact and that the leftmost bottom vertical arrow is surjective. The exactness of the middle row follows from the above discussion and the exactness of the bottom row follows from the definitions. The middle column is exact by Theorem 2.1; Theorem 2.1 and Proposition 2.8 together imply that the leftmost column is exact.

Consider the map induced by $\beta \circ j$

$$\frac{\left(\operatorname{Pic}\overline{Y}/(\pi^*\operatorname{Pic} S + 2\operatorname{Pic}\overline{Y})\right)^{G_{k'}}}{\operatorname{Pic} Y_{k'}/(\pi^*\operatorname{Pic} S + 2\operatorname{Pic} Y_{k'})} \hookrightarrow \frac{\operatorname{Br}_1 U_{k'}}{\operatorname{Br} k'}[2];$$

we would like to show that it is surjective. Since $U = Y \setminus \bigcup_{E \in \mathcal{E}} \pi^{-1}(E)$ and the elements of $\mathcal{E}$ form an integral basis for $\operatorname{Pic} S = \operatorname{Pic}\overline{S}$, we have that $\operatorname{Pic}\overline{U} \cong \operatorname{Pic}\overline{Y}/\pi^*\operatorname{Pic} S$ and $\operatorname{Pic} U_{k'} \cong \operatorname{Pic} Y_{k'}/\pi^*\operatorname{Pic} S$. In particular,

$$\frac{\operatorname{Pic}\overline{Y}}{\pi^*\operatorname{Pic} S + 2\operatorname{Pic}\overline{Y}} \xrightarrow{\sim} \frac{\operatorname{Pic}\overline{U}}{2\operatorname{Pic}\overline{U}} \quad \text{and} \quad \frac{\operatorname{Pic} Y_{k'}}{\pi^*\operatorname{Pic} S + 2\operatorname{Pic} Y_{k'}} \xrightarrow{\sim} \frac{\operatorname{Pic} U_{k'}}{2\operatorname{Pic} U_{k'}}.$$

Furthermore, since $\operatorname{Pic}\overline{U}[2] = 0$, the cohomological long exact sequence associated to the multiplication by 2 map yields the isomorphism

$$\frac{(\operatorname{Pic}\overline{U}/2\operatorname{Pic}\overline{U})^{G_{k'}}}{(\operatorname{Pic}\overline{U})^{G_{k'}}/(2\operatorname{Pic}\overline{U})^{G_{k'}}} \cong \mathrm{H}^1(G_{k'}, \operatorname{Pic}\overline{U})[2].$$

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In addition, since $\mathrm{H}^0(\overline{U}, \mathbb{G}_m) = \overline{k}^\times$, the long exact sequence of low degree terms from the Hochschild-Serre spectral sequence [Mil80, Thm. 2.20] gives the isomorphism

$$\frac{Br_1 U_{k'}}{\mathrm{Br}\, k'} \xrightarrow{\sim} \mathrm{H}^1(G_{k'}, \mathrm{Pic}\,\overline{U}).$$

Since all groups in question are finite, a cardinality argument completes the proof of surjectivity, and hence the proof of the theorem. $\qquad\square$

3. Geometry of $X_\mathbf{a}$ and $Y_\mathbf{a}$

3.1. **Review of** [VAV11, §4]**.** The K3 surface $Y_\mathbf{a}$ is defined as the base locus of a net of quadrics. As explained in [Bea96, Example IX.4.5] and [VAV11, §4.1], each isolated singular point in the degeneracy locus of the net gives rise to two genus 1 fibrations on $\overline{Y}_\mathbf{a}$. As the degeneracy locus of the net has 14 singular points, we obtain 28 classes of curves in $\mathrm{Pic}\,\overline{Y_\mathbf{a}}$, which we denote $F_1, G_1, \ldots, F_{14}, G_{14}$; for all i, j we have the relation $F_i + G_i = F_j + G_j$.

Nine of these singular points define fibrations which descend to $\overline{X}_\mathbf{a}$. On $\overline{X}_\mathbf{a}$, these fibrations have exactly two nonreduced fibers. For a fixed point P_i in the degeneracy locus, we let $C_i, \widetilde{C}_i, D_i$, and $\widetilde{D}_i$ denote the reduced subschemes of the nonreduced fibers of the corresponding fibrations. After possibly relabeling, we may assume that we have the following relations in $\mathrm{Pic}\,\overline{X}_\mathbf{a}$:

$$C_i + D_i = \widetilde{C}_j + \widetilde{D}_j, \quad 2(C_i - \widetilde{C}_i) = 2(D_i - \widetilde{D}_i) = 0, \quad f^* C_i = f^* \widetilde{C}_i = F_i, \quad f^* D_i = f^* \widetilde{D}_i = G_i.$$

Defining equations for curves representing each of these classes is given in Table A.4. The intersection numbers of these curves are as follows:

$$\begin{array}{llll}
F_i^2 = G_i^2 = 0, & F_i \cdot G_i = 4, & F_i \cdot G_j = F_i \cdot F_j = G_i \cdot G_j = 2 \text{ for all } i \neq j, \\
C_i^2 = D_i^2 = 0, & C_i \cdot D_i = 2, & C_i \cdot D_j = C_i \cdot C_j = D_i \cdot D_j = 1 \text{ for all } i \neq j.
\end{array}$$

Proposition 3.1 ([VAV11, Cor 4.3])**.** *Let* $\mathbf{a} \in \mathbb{Z}_{>0}^3$ *satisfy conditions* (5)*,* (6)*, and* (8)*. Then* $\mathrm{Pic}\,\overline{Y}_\mathbf{a} \cong \mathbb{Z}^{15}$ *and is generated by* $G_1, F_1, \ldots, F_{14}$ *and*

$$Z_1 := \frac{1}{2}\left(F_1 + F_2 + F_3 + F_{10} + F_{12}\right), \quad Z_2 := \frac{1}{2}\left(F_1 + G_1 + F_4 + F_5 + F_6 + F_{10} + F_{11}\right),$$

$$Z_3 := \frac{1}{2}\left(F_1 + F_4 + F_7 + F_{13} + F_{14}\right), \quad Z_4 := \frac{1}{2}\left(F_1 + G_1 + F_7 + F_8 + F_9 + F_{11} + F_{12}\right).$$

As in [VAV11], we let $K/\mathbb{Q}$ denote the splitting field of the curves F_i, G_i. We will be concerned with two particular subfields $K_0 \subset K_1 \subset K$; we give generators for these fields in Appendix 5 and describe their defining properties in §§3.2–3.3.

3.2. **Double cover morphisms.** In order to apply the results of §2, we must realize $Y_\mathbf{a}$ as a double cover of a rational ruled surface. We will be able to do so over the Galois extension $K_0 := \mathbb{Q}(i, \sqrt{2}, \sqrt{5}, \sqrt{-2 + 2\sqrt{2}})$; throughout this section, we work over K_0.

Consider the morphism

$$\phi \colon Y_\mathbf{a} \to S^0 := \mathbb{P}_x^1 \times \mathbb{P}_t^1,$$

which sends a point $[v_0 : v_1 : v_2 : w_0 : w_1 : w_2]$ to

$$\left(\left[w_0 - \sqrt{5} w_1 : v_0 + 2v_1 \right], \left[\sqrt{-2 + 2\sqrt{2}} w_0 - \sqrt{5} w_1, v_0 + \sqrt{2} v_1 + \sqrt{5}(1 - \sqrt{2}) v_2 \right] \right). \quad (3.1)$$

For any $P \in Y_{\mathbf{a}}$, we have $\phi(\sigma(P)) = [-1](\phi(P))$, where $[-1]$ denotes the automorphism of S^0 that sends (x, t) to $(-x, -t)$. Thus, we have an induced morphism

$$\tilde{\phi} \colon X_{\mathbf{a}} \to \left(S^0\right)/[-1],$$

obtained by quotienting $Y_{\mathbf{a}}$ by σ and S^0 by $[-1]$.

The morphism ϕ factors through a double cover morphism $\pi \colon Y_{\mathbf{a}} \to S$, where $S :=$ $Y_{\mathbf{a}}/(w_2 \mapsto -w_2)$. The quotient S is a smooth del Pezzo surface of degree 4 given by

$$\left\{ v_0 v_1 + 5v_2^2 - w_0^2 = v_0^2 + 3v_0 v_1 + 2v_1^2 - w_0^2 + 5w_1^2 = 0 \right\} \subset \mathbb{P}^4_{(v_0:v_1:v_2:w_0:w_1)}.$$

Using (3.1), one can check that ϕ induces a birational map $\nu_S \colon S \to S^0$ which contracts four (-1)-curves; we denote these curves by E_1, E_2, E_3, and E_4. (Defining equations for the curves are given in Table A.3.)

The preimages of the E_i under π are irreducible (-2)-curves in $Y_{\mathbf{a}}$. Thus, we may also factor ϕ by first blowing-down these four (-2)-curves to obtain a (singular) surface $Y_{\mathbf{a}}^0$ and then quotienting by an involution to obtain a double cover morphism. Hence, we have the following commutative diagram,

$$\begin{array}{ccc} Y_{\mathbf{a}} & \xrightarrow{\pi} & S \\ \downarrow{\scriptstyle \nu} & \searrow{\scriptstyle \phi} & \downarrow{\scriptstyle \nu_S} \\ Y_{\mathbf{a}}^0 & \xrightarrow{\pi^0} & S^0 \end{array}$$

where the vertical maps are birational and the horizontal maps are 2-to-1. Note that over $k = K_0$, these varieties and morphisms satisfy all assumptions from §2. In particular, all morphisms are defined over K_0 and $\operatorname{Pic} S_{K_0} = \operatorname{Pic} \overline{S}$.

Since σ induces an involution on $Y_{\mathbf{a}}^0$, S, and S^0 which is compatible with the morphisms, the above commutative diagram descends to the following commutative diagram involving the Enriques surface:

$$\begin{array}{ccc} X_{\mathbf{a}} & \xrightarrow{\tilde{\pi}} & \tilde{S} \\ \downarrow{\scriptstyle \tilde{\nu}} & \searrow{\scriptstyle \tilde{\phi}} & \downarrow{\scriptstyle \tilde{\nu}_S} \\ X_{\mathbf{a}}^0 & \xrightarrow{\tilde{\pi}^0} & (S^0)/[-1] \end{array}$$

3.3. **Branch loci of the double covers.** Let $B, B^0, \widetilde{B}, \widetilde{B}^0$ denote the branch loci of $\pi, \pi^0, \tilde{\pi}$ and $\tilde{\pi}^0$ respectively.

Proposition 3.2.
(1) $B = V(w_2) \subset \mathbb{P}^4$ is a smooth genus 5 curve,
(2) $\widetilde{B}$ is a smooth genus 3 curve,
(3) B^0 is an arithmetic genus 9 curve with 4 nodal singularities, and
(4) $\widetilde{B}^0$ is an arithmetic genus 5 curve with 2 nodal singularities.

Furthermore, the projection morphism $\widetilde{B} \to V(av_0^2 + bv_1^2 + cv_2^2) \subset \mathbb{P}^2$ is a double cover map, so $\widetilde{B}$ is geometrically hyperelliptic.

Proof. From the definition of π, one can immediately see that B is the image of $V(w_2) \cap Y_{\mathbf{a}}$, so B is given by an intersection of three quadrics in $\mathbb{P}^4$. Since (1.1) holds, B is smooth by the Jacobian criterion. Therefore, B is a smooth genus 5 curve.

As $B/\sigma = \widetilde{B}$ and $\sigma|_B$ has no fixed points, the Riemann-Hurwitz formula implies that $\widetilde{B}$ is a smooth genus 3 curve. Furthermore, the curve B has a 4-to-1 projection map to the plane conic $\{av_0^2 + bv_1^2 + cv_2^2 = 0\} \subset \mathbb{P}^2$. This induces a double cover map from $\widetilde{B}$ to the same conic, so $\widetilde{B}$ is (geometrically) hyperelliptic.

Using the equations given in Table A.3 and computer algebra software, one can check that each (-1)-curve E_i intersects B transversely in 2 distinct points. Thus, the curve B^0 has four nodal singularities, and so has arithmetic genus 9. As $B^0 \to \widetilde{B}^0$ is an étale double cover, B^0 has two nodal singularities and the Riemann-Hurwitz formula implies that $\widetilde{B}^0$ has arithmetic genus 5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since $\widetilde{B}$ is geometrically hyperelliptic, we may identify $\mathrm{Jac}(\widetilde{B}_{\overline{\mathbb{Q}}})[2]$ with subsets of the Weierstrass points of $\widetilde{B}$ of even order, modulo identifying complementary subsets. Under this identification, the group operation is given by the symmetric difference, i.e., $A + B = (A \cup B) \setminus (A \cap B)$. One can easily see that the ramification locus of the projection morphism $B \to V(av_0^2 + bv_1^2 + cv_2^2) \subset \mathbb{P}^2$ is given by $V(w_0) \cup V(w_1) \subset B$. Since this projection morphism factors as

$$B \longrightarrow \widetilde{B} \longrightarrow V(av_0^2 + bv_1^2 + cv_2^2) \subset \mathbb{P}^2,$$

where the first morphism is étale, the Weierstrass points of $\widetilde{B}$ are $f(V(w_0)) \cup f(V(w_1))$. There are four points in $f(V(w_0)) \subset \widetilde{B}$, which we denote by P_1, P_2, P_3 and P_4, and four points in $f(V(w_1)) \subset \widetilde{B}$, which we denote by Q_1, Q_2, Q_3, and Q_4. We let

$$K_1/K_0 := \text{the splitting field of the Weierstrass points.}$$

Proposition 3.3.

(1) *The kernel of $f^*\colon \mathrm{Jac}(\widetilde{B}_{\overline{\mathbb{Q}}})[2] \to \mathrm{Jac}(B_{\overline{\mathbb{Q}}})[2]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and the unique nontrivial element is $\{P_1, P_2, P_3, P_4\}$.*

(2) *The image of f^* fits in the following* non-split *exact sequence of G_{K_0}-modules:*

$$0 \to \mathrm{Ind}_{K_0(\theta_0)}^{K_0}(\mathbb{Z}/2\mathbb{Z}) \times \mathrm{Ind}_{K_0(\sqrt{ab})}^{K_0}(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{im}\, f^* \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Proof. **(1)** Let P_i' and P_i'' be the inverse image of P_i under the map $B \to \widetilde{B}$. The vanishing locus $V(w_0)$ on B consists of P_i', P_i'' for $i = 1, \ldots, 4$. For any distinct pair of numbers $i_0, i_1 \in \{1, 2, 3, 4\}$, there exists a linear form $L = L(v_0, v_1, v_2)$ such that $V(L) \subset \mathbb{P}^2$ contains the images of P_{i_0} and P_{i_1}. Since the points P_i are Weierstrass points of $\widetilde{B}$ and the map $B \to \widetilde{B}$ is étale, this implies that

$$\mathrm{div}_B(w_0/L) = \sum_{i=1}^{4}(P_i + P_i') - 2(P_{i_0}' + P_{i_0}'' + P_{i_1}' + P_{i_1}'').$$

Thus $f^*\{P_1, P_2, P_3, P_4\}$ is principal on B.

Now let D be a divisor such that $[D] \in \mathrm{Jac}\,\widetilde{B}[2]$ is a nontrivial element of $\ker f^*$. Then $f^{-1}(D) = \mathrm{div}_B(g_0)$, for some function $g_0 \in \mathbf{k}(B)$. On the other hand, $2D = \mathrm{div}_{\widetilde{B}}(g_1)$ for some function $g_1 \in \mathbf{k}(\widetilde{B})$. Thus, $g_1 = g_0^2$ in $\mathbf{k}(B)^\times/k^\times$. Since $\mathbf{k}(B)$ is a quadratic extension of $\mathbf{k}(\widetilde{B})$ generated by w_0/L for $L = L(v_0, v_1, v_2)$ a linear form, this implies that $g_0 = h \cdot (w_0/L)$ in $\mathbf{k}(B)$ for some $h \in \mathbf{k}(\widetilde{B})^{\times 2}$. (Note that $w_0/L \notin \mathbf{k}(\widetilde{B})$.) Hence $D = \{P_1, P_2, P_3, P_4\}$ in $\mathrm{Jac}\,\widetilde{B}$.

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(2) By (1) and the description of Jac $\widetilde{B}[2]$ in terms of Weierstrass points, we see that the image of f^* is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$ (as a group) and is generated by

$$\{P_1, P_3\}, \ \{P_1, P_4\}, \ \{Q_1, Q_3\}, \ \{Q_1, Q_4\}, \ \text{and} \ \{P_1, Q_1\}.$$

Using Table A.5, one can check that (as G_{K_0}-modules) we have:

$$\langle\{P_1, P_3\}, \ \{P_1, P_4\}\rangle \cong \mathrm{Ind}_{K_0(\sqrt{ab})}^{K_0}(\mathbb{Z}/2\mathbb{Z}), \quad \text{and} \quad \langle\{Q_1, Q_3\}, \ \{Q_1, Q_4\}\rangle \cong \mathrm{Ind}_{K_0(\theta_0)}^{K_0}(\mathbb{Z}/2\mathbb{Z}),$$

thus proving the existence of the exact sequence. Since $\mathrm{Gal}(K_1/K_0)$ acts transitively on $\{P_1, P_2, P_3, P_4\}$, no element of the form $\{P_i, Q_j\}$ is fixed by G_{K_0} and so the exact sequence does not split. $\qquad\square$

3.4. **The quotient group** $\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/\left(\pi^*\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}\right)$. The following lemma about the structure of $\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/\left(\pi^*\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}\right)$ as a Galois module will be useful in later sections.

Lemma 3.4. *We have an isomorphism of* $\mathrm{Gal}(K_1/K_0)$-*modules*

$$\left(\frac{\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}}{\pi^*\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}}\right)^{G_{K_1}} \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \times \mathrm{Ind}_{K_0(\theta_0)}^{K_0}(\mathbb{Z}/2\mathbb{Z}) \times \mathrm{Ind}_{K_0(\sqrt{ab})}^{K_0}(\mathbb{Z}/2\mathbb{Z}).$$

Proof. By computing intersection numbers, we find that $\pi^*\mathrm{Pic}\,S$ is generated by

$$G_1, \ F_1 - F_2, \ Z_1 - F_2 - F_{10} - F_{12}, \ Z_1 - F_1 - F_2, \ -Z_1 + F_1 + F_{12}, \ \text{and} \ -Z_1 + F_1 + F_{10}.$$

Therefore, $\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/\left(\pi^*\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}\right)$ is a 9-dimensional $\mathbb{F}_2$ vector space with basis

$$F_5, \ F_6, \ F_8, \ F_9, \ F_{11}, \ F_{13}, \ Z_2, \ Z_3, \ \text{and} \ Z_4.$$

In this quotient we have the relations $G_i = F_i$, $F_1 = F_2 = F_{10} = F_{12} = Z_1 = G_1 = F_3 = 0$,

$$F_4 = F_5 + F_6 + F_{11}, \quad F_7 = F_8 + F_9 + F_{11}, \quad \text{and} \quad F_{14} = F_5 + F_6 + F_8 + F_9 + F_{13}.$$

Using this basis and Table A.5, we compute that $(\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/\left(\pi^*\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}\right))^{G_{K_1}}$ is a 5-dimensional $\mathbb{F}_2$-vector space with basis F_5, F_6, F_8, F_9, and F_{11}. The isomorphism then follows after noting that in the quotient F_{11} is fixed by $\mathrm{Gal}(K_1/K_0)$, F_5 and F_6 are interchanged by $\mathrm{Gal}(K_0(\sqrt{ab})/K_0)$ and fixed by all other elements, and similarly for F_8 and F_9 and $\mathrm{Gal}(K_0(\theta_0)/K_0)$. $\qquad\square$

4. A representative of the nontrivial Brauer class on $\overline{X}_{\mathbf{a}}$

As mentioned in the introduction, $\mathrm{Br}\,\overline{X}_{\mathbf{a}} \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, there is at most one non-trivial class in $\mathrm{Br}\,X_{\mathbf{a}}/\mathrm{Br}_1\,X_{\mathbf{a}}$. To determine the existence of such a class, we must first obtain an explicit representative for the unique Brauer class in $\mathrm{Br}\,\overline{X}_{\mathbf{a}}$. Using Beauville's criterion [Bea09, Cor. 5.7], Várilly-Alvarado and the last author showed that if $\mathbf{a}$ satisfies conditions (5), (6) and (8), then

$$\ker\left(f^* \colon \mathrm{Br}\,\overline{X}_{\mathbf{a}} \to \mathrm{Br}\,\overline{Y}_{\mathbf{a}}\right) = 0. \quad [\text{VAV11, Proof of Prop. 5.2}]$$

Let D be a divisor on $\widetilde{B}_{\overline{\mathbb{Q}}}$ such that $[D] \in \mathrm{Jac}(\widetilde{B})[2]$ and such that $[D]$ corresponds to a subset of the Weierstrass points containing an *odd* number of the points $\{P_i\}_{i=1,\ldots,4}$. Let $\tilde{\ell} \in \mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})$ be such that $\mathrm{div}(\tilde{\ell}) = 2D$.

Proposition 4.1. *Assume that* $\mathbf{a}$ *satisfies conditions* (5), (6), *and* (8). *Then the Brauer class*

$$\pi^* \mathcal{A}_{\tilde{\ell}} = \pi^* \mathrm{Cor}_{\mathbf{k}(\overline{B})(x)/\overline{\mathbb{Q}}(t,x)} \left((\tilde{\ell}, x - \alpha)_{\overline{\mathbb{Q}}} \right)$$

defines an element of $\mathrm{Br}\, \overline{Y}_{\mathbf{a}}$ *and generates the order* 2 *subgroup* $f^* \mathrm{Br}\, \overline{X}_{\mathbf{a}}$.

Proof. By [CV15, Thm 7.2], the subgroup $f^* \mathrm{Br}\, X_{\mathbf{a}}[2] \subset \mathrm{Br}\, \overline{Y}_{\mathbf{a}}$ is generated by $\mathcal{A}_{\tilde{\ell}'}$, where $\tilde{\ell}' \in \mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})$ is such that $\nu_*(\mathrm{div}(\tilde{\ell}')) \in 2\,\mathrm{Div}(\widetilde{B}^0)$. Furthermore, by Theorem 2.1 applied to $k' = \overline{k}$ we have the following exact sequence

$$0 \to \frac{\mathrm{Pic}\, \overline{Y}_{\mathbf{a}}}{\pi^* \mathrm{Pic}\, S + 2\,\mathrm{Pic}\, \overline{Y}} \xrightarrow{\;j\;} \frac{\mathbf{k}(B_{\overline{\mathbb{Q}}})^{\times}_{\mathcal{E}}}{\mathbf{k}(B_{\overline{\mathbb{Q}}})^{\times 2}} \longrightarrow \mathrm{Br}\, \overline{Y}_{\mathbf{a}}[2] \longrightarrow 0.$$

Thus, to prove the proposition, it remains to prove that $[\tilde{\ell}]$ is not in the image of j.

For convenience, we set:

$$\tilde{G} := \{\ell \in \mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})^{\times} : \nu_* \,\mathrm{div}(\ell) \in 2\,\mathrm{Div}(\widetilde{B}^0_{\overline{\mathbb{Q}}})\}.$$

We have the following commutative diagram with exact rows [CV15, Proposition 4.5], where $\mathrm{Sing}(B_{\overline{\mathbb{Q}}})$ and $\mathrm{Sing}(\widetilde{B}_{\overline{\mathbb{Q}}})$ denote the set of singular points of $B_{\overline{\mathbb{Q}}}$ and $\widetilde{B}_{\overline{\mathbb{Q}}}$, respectively, and the last vertical map is the diagonal embedding on the first factor, i.e., $(m, n) \mapsto (m, m, n, n, 0, 0)$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2] & \longrightarrow & \dfrac{\tilde{G}}{\mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})^{\times 2}} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Sing}(\widetilde{B}_{\overline{\mathbb{Q}}})} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Jac}\, B_{\overline{\mathbb{Q}}}[2] & \longrightarrow & \dfrac{\mathbf{k}(B_{\overline{\mathbb{Q}}})_{\mathcal{E}}}{\mathbf{k}(B_{\overline{\mathbb{Q}}})^{\times 2}} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Sing}(B_{\overline{\mathbb{Q}}})} \times (\mathbb{Z}/2\mathbb{Z})^2 & \longrightarrow & 0.
\end{array}
$$

In the rest of this section, we will view elements of $\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2]$ and $\mathrm{Jac}\, B_{\overline{\mathbb{Q}}}[2]$ as elements of $\tilde{G}/\mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})^{\times 2}$ and $\mathbf{k}(B_{\overline{\mathbb{Q}}})_{\mathcal{E}}/\mathbf{k}(B_{\overline{\mathbb{Q}}})^{\times 2}$, respectively.

Lemma 4.2. *Assume that* $\mathbf{a}$ *satisfies conditions* (5), (6), *and* (8). *Then the group* $\mathrm{im}\, j \cap f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2])$ *is an index* 2 $\mathrm{Gal}(\overline{\mathbb{Q}}/K_0)$-*invariant subgroup of* $f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2])$.

Proof. Since $\mathrm{Br}\, \overline{X}_{\mathbf{a}} \cong \mathbb{Z}/2\mathbb{Z}$, the subgroup $\mathrm{im}\, j \cap f^*(\tilde{G})$ has index 2 in $f^*(\tilde{G})$. Therefore, the subgroup $\mathrm{im}\, j \cap f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2])$ has index at most 2 in $f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2])$.

Recall that K_1 is splitting field of the Weierstrass points over K_0. Thus $\mathrm{Gal}(\overline{\mathbb{Q}}/K_1)$ fixes $f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}})$ and hence $\mathrm{im}\, j \cap f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2]) = (\mathrm{im}\, j)^{G_{K_1}} \cap f^*(\mathrm{Jac}\, \widetilde{B}_{\overline{\mathbb{Q}}}[2])$. Since j and f^* are G_{K_0}-equivariant homomorphisms, the intersection is G_{K_0}-invariant and the elements in the intersection must have compatible G_{K_0} action. Then Proposition 3.3 and Lemma 3.4 together imply that the intersection is a submodule of $\mathrm{Ind}^{K_0}_{K_0(\theta_0)}(\mathbb{Z}/2\mathbb{Z}) \times \mathrm{Ind}^{K_0}_{K_0(\sqrt{ab})}(\mathbb{Z}/2\mathbb{Z})$ and hence a proper subgroup of $f^*(\mathrm{Jac}\, \widetilde{B}[2])$ with index equal to 2. $\qquad\square$

Now we resume the proof of Proposition 4.1. Assume that $\tilde{\ell}$ is contained in the image of j and hence in $\mathrm{im}\, j \cap f^*(\mathrm{Jac}\, \widetilde{B}[2])$. From Tables A.1 and A.5, we see that $\mathrm{Gal}(\overline{\mathbb{Q}}/K_0)$ acts transitively on the Weierstrass points of $\widetilde{B}$. Therefore, the subgroup of $\mathbf{k}(B_{\overline{\mathbb{Q}}})_{\mathcal{E}}$ generated by

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$\widetilde{\ell}$ and all of its $\mathrm{Gal}(\overline{\mathbb{Q}}/K_0)$ conjugates contains all of $f^*(\mathrm{Jac}\,\widetilde{B}_{\overline{\mathbb{Q}}}[2])$. Since $\mathrm{im}\,j \cap f^*(\mathrm{Jac}\,\widetilde{B}[2])$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/K_0)$-invariant, this implies that

$$\mathrm{im}\,j \cap f^*(\mathrm{Jac}\,\widetilde{B}_{\overline{\mathbb{Q}}}[2]) = f^*(\mathrm{Jac}\,\widetilde{B}_{\overline{\mathbb{Q}}}[2])$$

which contradicts Lemma 4.2. $\qquad\square$

5. Proof of Theorem 1.2

Assume that $\mathbf{a} \in \mathbb{Z}_{>0}^3$ satisfies conditions $(5), (6)$, and (8). The last statement of the theorem follows immediately from the first statement together with [VAV11, Thm. 1.2]. Thus our goal is to prove that $\mathrm{Br}\,X_{\mathbf{a}} = \mathrm{Br}_1\,X_{\mathbf{a}}$. Since $\mathrm{Br}\,X_{\mathbf{a}} \to \mathrm{Br}\,\overline{X}_{\mathbf{a}}$ factors through $(\mathrm{Br}\,X_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)}$, it suffices to prove that $(\mathrm{Br}\,X_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)} = (\mathrm{Br}_1\,X_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)}$.

Recall from §4 that $f^*\,\mathrm{Br}\,\overline{X}_{\mathbf{a}}$ is a non-trivial subgroup of $\mathrm{Br}\,\overline{Y}_{\mathbf{a}}$. So if $(\mathrm{Br}\,X_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)}$ is strictly larger than $(\mathrm{Br}_1\,X_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)}$, then there exists an element $\mathcal{B} \in (\mathrm{Br}\,Y_{\mathbf{a},K_1})^{\mathrm{Gal}(K_1/K_0)}$ such that $\mathcal{B}_{\overline{\mathbb{Q}}}$ is the unique non-trivial element in $f^*\,\mathrm{Br}\,\overline{X}_{\mathbf{a}}$. Let $\widetilde{\ell} \in \mathbf{k}(\widetilde{B}_{\overline{\mathbb{Q}}})$ be such that $\mathrm{div}(\widetilde{\ell}) = 2D$ where $[D] \in \mathrm{Jac}(\widetilde{B})[2]$ corresponds to a subset of the Weierstrass points containing an odd number of the points P_i. Note that we may choose $\widetilde{\ell}$ so that it is contained in $\mathbf{k}(\widetilde{B}_{K_1})^{\times}$. In what follows, we will view $\widetilde{\ell}$ as a function on B under the natural inclusion $\mathbf{k}(\widetilde{B}) \subset \mathbf{k}(B)$. We remark that, under this inclusion, $\widetilde{\ell} \in \mathbf{k}(B_{K_1})^{\times}_{\mathcal{E}}$.

By Proposition 4.1, $f^*\,\mathrm{Br}\,\overline{X}_{\mathbf{a}}$ is generated by $(\pi^*\mathcal{A}_{\widetilde{\ell}})_{\overline{\mathbb{Q}}}$, so $\mathcal{B}_{\overline{\mathbb{Q}}} = (\pi^*\mathcal{A}_{\widetilde{\ell}})_{\overline{\mathbb{Q}}}$. Let $\mathcal{A}' := \mathcal{A}'(\widetilde{\ell})$ be as in Proposition 2.6. Then an application of Tsen's theorem shows $(\pi^*(\varpi^*\mathcal{A}'))_{\overline{\mathbb{Q}}} = 0$. Hence, $\beta(\widetilde{\ell})_{\overline{\mathbb{Q}}} = (\pi^*\mathcal{A}_{\widetilde{\ell}})_{\overline{\mathbb{Q}}} = \mathcal{B}_{\overline{\mathbb{Q}}}$. Also, since $\beta(\widetilde{\ell}) \in \mathrm{Br}\,U_{K_1}$, we must have that

$$\mathcal{B} - \beta(\widetilde{\ell}) \in \mathrm{Br}_1\,U_{K_1}.$$

By Theorem 2.2, $\mathrm{Br}_1\,U_{K_1}/\mathrm{Br}\,K_1$ is contained in $\beta(\mathbf{k}(B_{K_1})^{\times}_{\mathcal{E}})$, meaning that $\mathcal{B} - \beta(\widetilde{\ell}) = \beta(\ell')$ for some $\ell' \in \mathbf{k}(B_{K_1})^{\times}_{\mathcal{E}}$. Since both ℓ' and $\widetilde{\ell}$ are in $\mathbf{k}(B_{K_1})^{\times}_{\mathcal{E}}$, we then have $\mathcal{B} = \beta(\widetilde{\ell}\ell') =: \beta(\ell_{\mathcal{B}})$ for some $\ell_{\mathcal{B}} \in \mathbf{k}(B_{K_1})_{\mathcal{E}}$. Furthermore, since $\mathcal{B}$ is $\mathrm{Gal}(K_1/K_0)$-invariant and is equal to $\beta(\widetilde{\ell})$ modulo $\mathrm{Br}_1\,U_{K_1}$, Theorem 2.2 implies that

(a) the class of $\ell_{\mathcal{B}}$ in $\mathbf{k}(B_{K_1})/j(\mathrm{Pic}\,Y_{\mathbf{a},K_1})K_1^{\times}\mathbf{k}(B_{K_1})^{\times 2}$ must be $\mathrm{Gal}(K_1/K_0)$-invariant, and
(b) $\ell_{\mathcal{B}}\widetilde{\ell}^{-1} \in j\left((\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/(\pi^*\,\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}))^{G_{K_1}}\right)K_1^{\times}\mathbf{k}(B_{K_1})^{\times 2}$.

An inspection of Table A.5 reveals that $\mathrm{Pic}\,Y_{\mathbf{a},K_1} = \mathrm{Pic}\,S_{K_1}$, so $j(\mathrm{Pic}\,Y_{\mathbf{a},K_1}) \subset K_1^{\times}\mathbf{k}(B_{K_1})^{\times 2}$. In addition, Lemma 3.4 shows that every element of $j\left((\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}/(\pi^*\,\mathrm{Pic}\,S + 2\,\mathrm{Pic}\,\overline{Y}_{\mathbf{a}}))^{G_{K_1}}\right)$ is $\mathrm{Gal}(K_0(\theta_0, \sqrt{ab})/K_0)$-invariant. Thus, conditions (a) and (b) imply that

$$\widetilde{\ell} \in \left(\frac{\mathbf{k}(B_{K_1})^{\times}}{K_1^{\times}\mathbf{k}(B_{K_1})^{\times 2}}\right)^{\mathrm{Gal}(K_0(\theta_0, \sqrt{ab})/K_0)}.$$

Given our assumption on $\widetilde{\ell}$, this results in a contradiction, as demonstrated by Table A.5. $\quad\square$

APPENDIX A. FIELDS, DEFINING EQUATIONS, AND GALOIS ACTIONS

The splitting field K of the genus 1 curves $C_i, \widetilde{C}_i, \widetilde{D}_i, F_i, G_i$ is generated by

$$i, \sqrt{2}, \sqrt{5}, \sqrt{a}, \sqrt{c}, \eta_0 := \sqrt{c^2 - 100ab}, \gamma_0 := \sqrt{-c^2 - 5bc - 10ac - 25ab}, \tag{A.1}$$

$$\sqrt[4]{ab}, \sqrt{-2 + 2\sqrt{2}}, \sqrt{-c - 10\sqrt{ab}}, \tag{A.2}$$

$$\theta_0 := \sqrt{4a^2 + b^2}, \ \xi_0 := \sqrt{a + b + c/5}, \ \xi_0' := \sqrt{a + b/4 + c/10}, \tag{A.3}$$

$$\theta_1^+ := \sqrt{20a^2 - 10ab - 2bc + (10a + 2c)\theta_0}, \ \theta_2^+ := \sqrt{-5a - 5/2b - 5/2\theta_0}, \tag{A.4}$$

$$\xi_1^+ := \sqrt{20a + 10b + 3c + 20\xi_0\xi_0'}, \ \xi_2^+ := \sqrt{4a + 2b + 2/5c + 4\xi_0\xi_0'}. \tag{A.5}$$

Define

$$
\begin{aligned}
\eta_1^+ &:= \frac{c - \eta_0 + 10\sqrt{ab}}{10\sqrt{a}\sqrt{-c - 10\sqrt{ab}}} & \gamma_1^+ &:= (\theta_1^+)^{-1}\left(10a^2 - 5ab - bc + 2a\gamma_0 + (c + 5a)\theta_0\right) \\
\eta_1^- &:= \frac{c + \eta_0 + 10\sqrt{ab}}{10\sqrt{a}\sqrt{-c - 10\sqrt{ab}}} & \gamma_1^- &:= (\theta_1^+)^{-1}\left(10a^2 - 5ab - bc - 2a\gamma_0 + (c + 5a)\theta_0\right).
\end{aligned}
\tag{A.6}
$$

The following subfields of K are of particular interest:

$$\mathbb{Q} \subset K_0 := \mathbb{Q}\left(i, \sqrt{2}, \sqrt{5}, \sqrt{-2 + 2\sqrt{2}}\right) \subset K_1 := K_0\left(\theta_0, \sqrt{ab}, \eta_1^+, \gamma_1^+\right) \subset K. \tag{A.7}$$

The field extensions $K/\mathbb{Q}$ and $K_1/\mathbb{Q}$ are Galois, as the following relations show:

$$\sqrt{-2 - 2\sqrt{2}} = \frac{2i}{\sqrt{-2 + 2\sqrt{2}}}, \quad \sqrt{-c + 10\sqrt{ab}} = \frac{\eta_0}{\sqrt{-c - 10\sqrt{ab}}},$$

$$\theta_1^- := \sqrt{20a^2 - 10ab - 2bc + (10a + 2c)\theta_0} = \frac{4a\gamma_0}{\theta_1^+}, \quad \theta_2^- := \sqrt{-5a - 5/2b + 5/2\theta_0} = \frac{5\sqrt{ab}}{\theta_2^+},$$

$$\xi_1^- := \sqrt{20a + 10b + 3c - 20\xi_0\xi_0'} = \frac{\eta_0}{\xi_1^+}, \quad \xi_2^- := \sqrt{4a + 2b + 2/5c - 4\xi_0\xi_0'} = \frac{2\gamma_0}{5\xi_2^+},$$

$$(\gamma_1^+)^2 = 10a^2 - 5ab - bc + 2a\gamma_0, \quad (\eta_1^+)^2 = \frac{-c + \eta_0}{50a},$$

$$(\gamma_1^-)^2 = 10a^2 - 5ab - bc - 2a\gamma_0, \quad (\eta_1^-)^2 = \frac{-c - \eta_0}{50a},$$

$$\gamma_1^+\gamma_1^- = (5a + c)\theta_0, \quad \eta_1^+\eta_1^- = \frac{-1}{5a}\sqrt{ab}.$$

Tables A.1 and A.4 show that these fields have the properties claimed in §3.

Remark A.1. Tables A.1 and A.4 list defining equations of a subvariety of the $Y_{\mathbf{a}}$. The image of this subvariety gives the corresponding object in $X_{\mathbf{a}}$.

$\tilde{B}$	Defining equations	$\tilde{B}$	Defining equations
P_1	$10av_0 - (c-\eta_0)v_1, \quad v_2 - \eta_1^+ v_1$	Q_1	$(c+5a)v_0 + (c-\gamma_0)v_1, \quad (c+5a)v_2 - \gamma_1^+ v_1$
P_2	$10av_0 - (c-\eta_0)v_1, \quad v_2 + \eta_1^+ v_1$	Q_2	$(c+5a)v_0 + (c-\gamma_0)v_1, \quad (c+5a)v_2 + \gamma_1^+ v_1$
P_3	$10av_0 - (c+\eta_0)v_1, \quad v_2 - \eta_1^- v_1$	Q_3	$(c+5a)v_0 + (c+\gamma_0)v_1, \quad (c+5a)v_2 - \gamma_1^- v_1$
P_4	$10av_0 - (c+\eta_0)v_1, \quad v_2 + \eta_1^- v_1$	Q_4	$(c+5a)v_0 + (c+\gamma_0)v_1, \quad (c+5a)v_2 + \gamma_1^- v_1$

TABLE A.1. Defining equations for the Weierstrass points on $\tilde{B}$

$\nu_S(E_1)$	$\left[1 - \sqrt{2} - i\sqrt{-2 + 2\sqrt{2}} : 1\right], \quad \left[(i - 1 + i\sqrt{2})\sqrt{-2 + 2\sqrt{2}} : 2\sqrt{2}\right]$
$\nu_S(E_2)$	$\left[-1 + \sqrt{2} + i\sqrt{-2 + 2\sqrt{2}} : 1\right], \quad \left[(1 - i - i\sqrt{2})\sqrt{-2 + 2\sqrt{2}} : 2\sqrt{2}\right]$
$\nu_S(E_3)$	$\left[1 - \sqrt{2} + i\sqrt{-2 + 2\sqrt{2}} : 1\right], \quad \left[-(i + 1 + i\sqrt{2})\sqrt{-2 + 2\sqrt{2}} : 2\sqrt{2}\right]$
$\nu_S(E_4)$	$\left[-1 + \sqrt{2} - i\sqrt{-2 + 2\sqrt{2}} : 1\right], \quad \left[(i + 1 + i\sqrt{2})\sqrt{-2 + 2\sqrt{2}} : 2\sqrt{2}\right]$

TABLE A.2. Defining equation for the exceptional curves on S

$2\pi^* E_1$	$F_1 + 2G_1 - F_2 + F_3 - F_{10} - F_{12}$
$2\pi^* E_2$	$-F_1 - F_2 + F_3 + F_{10} + F_{12}$
$2\pi^* E_3$	$F_1 + 2G_1 - F_2 - F_3 - F_{10} + F_{12}$
$2\pi^* E_4$	$F_1 + 2G_1 - F_2 - F_3 + F_{10} - F_{12}$

TABLE A.3. Pullbacks of exceptional curves in terms of F_i, G_i

$\overline{Y}_\mathbf{a}$	$\overline{X}_\mathbf{a}$	Defining equations	$\overline{Y}_\mathbf{a}$	$\overline{X}_\mathbf{a}$	Defining equations
F_1	C_1	$w_0 - \sqrt{5}w_1, \quad v_0 + 2v_1$	F_4	C_4	$\sqrt{c}w_0 - \sqrt{5}w_2, \quad 10av_0 - (c+\sqrt{c^2-100ab})v_1$
	$\widetilde{C}_1$	$w_0 + \sqrt{5}w_1, \quad v_0 + v_1$		$\widetilde{C}_4$	$\sqrt{c}w_0 + \sqrt{5}w_2, \quad 10av_0 - (c-\sqrt{c^2-100ab})v_1$
G_1	D_1	$w_0 - \sqrt{5}w_1, \quad v_0 + v_1$	G_4	D_4	$\sqrt{c}w_0 - \sqrt{5}w_2, \quad 10av_0 - (c-\sqrt{c^2-100ab})v_1$
	$\widetilde{D}_1$	$w_0 + \sqrt{5}w_1, \quad v_0 + 2v_1$		$\widetilde{D}_4$	$\sqrt{c}w_0 + \sqrt{5}w_2, \quad 10av_0 - (c+\sqrt{c^2-100ab})v_1$
F_2	C_2	$\sqrt{-2+2\sqrt{2}}w_0 - \sqrt{5}w_1, \quad v_0 + \sqrt{2}v_1 + \sqrt{5}(1-\sqrt{2})v_2$	F_5	C_5	$i\sqrt{2}\sqrt[4]{ab}w_0 - w_2, \quad \sqrt{a}v_0 + \sqrt{b}v_1 + \sqrt{-c-10\sqrt{ab}}v_2$
	$\widetilde{C}_2$	$\sqrt{-2+2\sqrt{2}}w_0 + \sqrt{5}w_1, \quad v_0 + \sqrt{2}v_1 - \sqrt{5}(1-\sqrt{2})v_2$		$\widetilde{C}_5$	$i\sqrt{2}\sqrt[4]{ab}w_0 + w_2, \quad \sqrt{a}v_0 + \sqrt{b}v_1 - \sqrt{-c-10\sqrt{ab}}v_2$
G_2	D_2	$\sqrt{-2+2\sqrt{2}}w_0 - \sqrt{5}w_1, \quad v_0 + \sqrt{2}v_1 - \sqrt{5}(1-\sqrt{2})v_2$	G_5	D_5	$i\sqrt{2}\sqrt[4]{ab}w_0 - w_2, \quad \sqrt{a}v_0 + \sqrt{b}v_1 - \sqrt{-c-10\sqrt{ab}}v_2$
	$\widetilde{D}_2$	$\sqrt{-2+2\sqrt{2}}w_0 + \sqrt{5}w_1, \quad v_0 + \sqrt{2}v_1 + \sqrt{5}(1-\sqrt{2})v_2$		$\widetilde{D}_5$	$i\sqrt{2}\sqrt[4]{ab}w_0 + w_2, \quad \sqrt{a}v_0 + \sqrt{b}v_1 + \sqrt{-c-10\sqrt{ab}}v_2$
F_3	C_3	$\sqrt{-2-2\sqrt{2}}w_0 - \sqrt{5}w_1, \quad v_0 - \sqrt{2}v_1 + \sqrt{5}(1+\sqrt{2})v_2$	F_6	C_6	$\sqrt{2}\sqrt[4]{ab}w_0 + w_2, \quad \sqrt{a}v_0 - \sqrt{b}v_1 + \sqrt{-c+10\sqrt{ab}}v_2$
	$\widetilde{C}_3$	$\sqrt{-2-2\sqrt{2}}w_0 + \sqrt{5}w_1, \quad v_0 - \sqrt{2}v_1 - \sqrt{5}(1+\sqrt{2})v_2$		$\widetilde{C}_6$	$\sqrt{2}\sqrt[4]{ab}w_0 - w_2, \quad \sqrt{a}v_0 - \sqrt{b}v_1 - \sqrt{-c+10\sqrt{ab}}v_2$
G_3	D_3	$\sqrt{-2-2\sqrt{2}}w_0 - \sqrt{5}w_1, \quad v_0 - \sqrt{2}v_1 - \sqrt{5}(1+\sqrt{2})v_2$	G_6	D_6	$\sqrt{2}\sqrt[4]{ab}w_0 + w_2, \quad \sqrt{a}v_0 - \sqrt{b}v_1 - \sqrt{-c+10\sqrt{ab}}v_2$
	$\widetilde{D}_3$	$\sqrt{-2-2\sqrt{2}}w_0 + \sqrt{5}w_1, \quad v_0 - \sqrt{2}v_1 + \sqrt{5}(1+\sqrt{2})v_2$		$\widetilde{D}_6$	$\sqrt{2}\sqrt[4]{ab}w_0 - w_2, \quad \sqrt{a}v_0 - \sqrt{b}v_1 + \sqrt{-c+10\sqrt{ab}}v_2$

$\overline{Y}_\mathbf{a}$	$\overline{X}_\mathbf{a}$	Defining equations	$\overline{Y}_\mathbf{a}$	$\overline{X}_\mathbf{a}$	Defining equations
F_7	C_7	$\sqrt{c}w_1 - w_2, \quad (5a+c)v_0 + (c+\gamma_0)v_1$	F_{10}		$w_0 - \sqrt{5}v_2, \quad v_0$
	$\widetilde{C}_7$	$\sqrt{c}w_1 + w_2, \quad (5a+c)v_0 + (c-\gamma_0)v_1$	G_{10}		$w_0 - \sqrt{5}v_2, \quad v_1$
G_7	D_7	$\sqrt{c}w_1 - w_2, \quad (5a+c)v_0 + (c-\gamma_0)v_1$	F_{11}		$w_2 - \sqrt{c}v_2, \quad \sqrt{a}v_0 + i\sqrt{b}v_1$
	$\widetilde{D}_7$	$\sqrt{c}w_1 + w_2, \quad (5a+c)v_0 + (c+\gamma_0)v_1$	G_{11}		$w_2 - \sqrt{c}v_2, \quad \sqrt{a}v_0 - i\sqrt{b}v_1$
F_8	C_8	$\theta_2^+ w_1 - w_2, \quad v_0 + (2a+b+\theta_0)/(b+\theta_0)v_1 - \theta_1^+/(2a)v_2$	F_{12}		$w_1 - v_2, \quad v_0 + (1-i)v_1$
	$\widetilde{C}_8$	$\theta_2^+ w_1 + w_2, \quad v_0 + (2a+b+\theta_0)/(b+\theta_0)v_1 + \theta_1^+/(2a)v_2$	G_{12}		$w_1 - v_2, \quad v_0 + (1+i)v_1$
G_8	D_8	$\theta_2^+ w_1 - w_2, \quad v_0 + (2a+b+\theta_0)/(b+\theta_0)v_1 + \theta_1^+/(2a)v_2$	F_{13}		$\xi_2^+ w_0 - \xi_1^+ w_1, \quad (\xi_0 + 2\xi_0')v_0 + (2\xi_0 + 2\xi_0')v_1 - w_2$
	$\widetilde{D}_8$	$\theta_2^+ w_1 + w_2, \quad v_0 + (2a+b+\theta_0)/(b+\theta_0)v_1 - \theta_1^+/(2a)v_2$	G_{13}		$\xi_2^+ w_0 - \xi_1^+ w_1, \quad (\xi_0 + 2\xi_0')v_0 + (2\xi_0 + 2\xi_0')v_1 + w_2$
F_9	C_9	$\theta_2^- w_1 - w_2, \quad v_0 + (2a+b-\theta_0)/(b-\theta_0)v_1 - \theta_1^-/(2a)v_2$	F_{14}		$\xi_2^- w_0 - \xi_1^- w_1, \quad (\xi_0 - 2\xi_0')v_0 + (2\xi_0 - 2\xi_0')v_1 - w_2$
	$\widetilde{C}_9$	$\theta_2^- w_1 + w_2, \quad v_0 + (2a+b-\theta_0)/(b-\theta_0)v_1 + \theta_1^-/(2a)v_2$	G_{14}		$\xi_2^- w_0 - \xi_1^- w_1, \quad (\xi_0 - 2\xi_0')v_0 + (2\xi_0 - 2\xi_0')v_1 + w_2$
G_9	D_9	$\theta_2^- w_1 - w_2, \quad v_0 + (2a+b-\theta_0)/(b-\theta_0)v_1 + \theta_1^-/(2a)v_2$			
	$\widetilde{D}_9$	$\theta_2^- w_1 + w_2, \quad v_0 + (2a+b-\theta_0)/(b-\theta_0)v_1 - \theta_1^-/(2a)v_2$			

TABLE A.4. Defining equations of a curve representing a divisor class.

Action on splitting field	Action on Pic $\overline{X}_\mathbf{a}$	Action on Pic $\overline{Y}_\mathbf{a}$	Action on Weierstrass points	Action on splitting field	Action on Pic $\overline{X}_\mathbf{a}$	Action on Pic $\overline{Y}_\mathbf{a}$	Action on Weierstrass points
$\sqrt{5} \mapsto -\sqrt{5}$	$C_1 \mapsto D_1$ $D_1 \leftrightarrow \widetilde{C}_1$ $C_2 \mapsto \widetilde{C}_2$ $C_3 \mapsto \widetilde{C}_3$ $C_4 \mapsto \widetilde{D}_4$	$F_1 \leftrightarrow G_1$ $F_4 \mapsto G_4$ $F_{10} \mapsto G_{10}$		$i \mapsto -i$	$C_3 \mapsto \widetilde{D}_3$ $C_5 \mapsto \widetilde{D}_5$	$F_3 \mapsto G_3$ $F_5 \mapsto G_5$ $F_{11} \mapsto G_{11}$ $F_{12} \mapsto G_{12}$	
				$\sqrt{-2+2\sqrt{2}}$ $\mapsto -\sqrt{-2+2\sqrt{2}}$	$C_2 \mapsto \widetilde{D}_2$ $C_3 \mapsto \widetilde{D}_3$	$F_2 \mapsto G_2$ $F_3 \mapsto G_3$	
$\sqrt{2} \mapsto -\sqrt{2}$ $\sqrt{-2+2\sqrt{2}}$ $\mapsto \sqrt{-2-2\sqrt{2}}$	$C_2 \leftrightarrow C_3$ $C_5 \mapsto \widetilde{D}_5$ $C_6 \mapsto \widetilde{D}_6$	$F_2 \leftrightarrow F_3$ $F_5 \mapsto G_5$ $F_6 \mapsto G_6$		$\sqrt{c} \mapsto -\sqrt{c}$	$C_4 \mapsto \widetilde{D}_4$ $C_7 \mapsto \widetilde{D}_7$	$F_4 \mapsto G_4$ $F_7 \mapsto G_7$ $F_{11} \mapsto G_{11}$	
$\gamma_0 \mapsto -\gamma_0$	$C_7 \mapsto D_7$ $C_9 \mapsto D_9$	$F_7 \mapsto G_7$ $F_9 \mapsto G_9$ $F_{14} \mapsto G_{14}$	$Q_1 \leftrightarrow Q_3$ $Q_2 \leftrightarrow Q_4$	$\eta_0 \mapsto -\eta_0$	$C_4 \mapsto D_4$ $C_6 \mapsto D_6$	$F_4 \mapsto G_4$ $F_6 \mapsto G_6$ $F_{14} \mapsto G_{14}$	$P_1 \leftrightarrow P_3$ $P_2 \leftrightarrow P_4$
$\sqrt[4]{ab} \mapsto i\sqrt[4]{ab}$ $\sqrt{-c-10\sqrt{ab}}$ $\mapsto \sqrt{-c+10\sqrt{ab}}$	$C_5 \mapsto C_6$ $C_6 \mapsto \widetilde{D}_5$ $C_9 \mapsto \widetilde{D}_9$	$F_5 \mapsto G_6$ $F_6 \mapsto G_5$ $F_9 \mapsto G_9$ $F_{11} \mapsto G_{11}$	$P_3 \leftrightarrow P_4$	$\sqrt{a} \mapsto -\sqrt{a}$	$C_5 \mapsto D_5$ $C_6 \mapsto D_6$	$F_5 \mapsto G_5$ $F_6 \mapsto G_6$	$P_1 \leftrightarrow P_2$ $P_3 \leftrightarrow P_4$
				$\sqrt{-c-10\sqrt{ab}}$ $\mapsto -\sqrt{-c-10\sqrt{ab}}$	$C_5 \mapsto D_5$ $C_6 \mapsto D_6$	$F_5 \mapsto G_5$ $F_6 \mapsto G_6$	
$\xi_0 \mapsto -\xi_0$ $\xi_i^+ \mapsto \xi_i^-$		$F_{13} \mapsto G_{14}$ $F_{14} \mapsto G_{13}$		$\theta_0 \mapsto -\theta_0$ $\theta_i^+ \mapsto \theta_i^-$	$C_8 \leftrightarrow C_9$	$F_8 \leftrightarrow F_9$	$Q_3 \leftrightarrow Q_4$
$\xi_0' \mapsto -\xi_0'$ $\xi_i^+ \mapsto \xi_i^-$		$F_{13} \leftrightarrow F_{14}$		$\theta_1^+ \mapsto -\theta_1^+$	$C_8 \mapsto D_8$ $C_9 \mapsto D_9$	$F_8 \mapsto G_8$ $F_9 \mapsto G_9$	$Q_1 \leftrightarrow Q_2$ $Q_3 \leftrightarrow Q_4$
$\xi_1^+ \mapsto -\xi_1^+$		$F_{13} \mapsto G_{13}$ $F_{14} \mapsto G_{14}$		$\theta_2^+ \mapsto -\theta_2^+$	$C_8 \mapsto \widetilde{D}_8$ $C_9 \mapsto \widetilde{D}_9$	$F_8 \mapsto G_8$ $F_9 \mapsto G_9$	
$\xi_2^+ \mapsto -\xi_2^+$		$F_{13} \mapsto G_{13}$ $F_{14} \mapsto G_{14}$					

TABLE A.5. The Galois action on the fibers of the genus 1 fibrations and the Weierstrass points.
The action on the splitting field is described by the action on the generators listed in A.1, A.2, A.3, A.4, A.5. If a generator of K is not listed, then we assume that it is fixed. We use the same convention for the curve classes and the Weierstrass points.

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