QUADRATIC POINTS ON INTERSECTIONS OF TWO QUADRICS

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Abstract. We prove that a smooth complete intersection of two quadrics of dimension at least 2 over a number field has index dividing 2, i.e., that it possesses a rational 0-cycle of degree 2.

1. Introduction

The index of a variety over a field k is the greatest common divisor of the degrees $[\mathbf{k}(x) : k]$ ranging over the residue fields $\mathbf{k}(x)$ of the (zero-dimensional) closed points x of the variety. Equivalently, the index is the smallest positive degree of a k-rational 0-cycle.

Let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over a field k of characteristic not equal to 2. Then the index of X necessarily divides 4, because intersecting with a plane yields a 0-cycle of degree 4. In general, this is the best possible bound. Indeed, there are examples with index 4 over local and global fields when $n = 3$ [LT58, Theorem 7] and over fields of characteristic 0 when $n = 4$, as we show in Theorem 7.6.

Our main result is the following sharp bound on the index when $n \geq 4$ and k is a number field or a local field.

Theorem 1.1. *Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n_k$ with $n \geq 4$ and assume that k is either a number field or a local field of characteristic not equal to 2. Then the index of X divides 2.*

This result allows us to complete the list of integers which occur as the index of a del Pezzo surface over a local field or a number field (See Section 7.4). It also allows us to deduce nontrivial index bounds for other interesting classes of varieties. If C/k is a genus 2 curve over a number field with a rational Weierstrass point, then it follows from the result above that any torsor of period 2 under the Jacobian of C has index dividing 8 (see Theorem 7.7) and the corresponding Kummer variety, which is an intersection of 3 quadrics in $\mathbb{P}^5$, has index dividing 4. Again, these results do not hold over arbitrary fields (see Remark 7.8).

Theorems of Amer, Brumer and Springer [Ame76, Bru78, Spr56] show that, for X as above, index 1 is equivalent to the existence of a k-rational point. Analogously one can ask if index 2 implies the existence of a closed point of degree 2. Colliot-Thélène has recently sketched an argument that if X is a smooth complete intersections of two quadrics in $\mathbb{P}^4$ over a field of characteristic 0 and X has index 2, then X has a closed point of degree $14, 6$ or 2. In this direction we prove the following.

Theorem 1.2. *Let $n \geq 4$ and let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over a field k of characteristic not equal to 2. In any of the following cases there is a quadratic extension K/k such that $X(K) \neq \emptyset$:*

(1) k is a local field;

(2) *k is a number field and Schinzel's hypothesis holds or k is a global field and Brauer-Manin is the only obstruction to the Hasse principle for del Pezzo surfaces of degree 4 over quadratic extensions of k, and one of the following holds:*
   (a) *$n \geq 5$;*
   (b) *$n = 4$, and for any quadratic field extension L/k and rank 4 quadric $\mathcal{Q} \subset \mathbb{P}^4_L$ such that $X = \cap_{\sigma \in \mathrm{Gal}(L/k)} \sigma(\mathcal{Q})$ and $\mathrm{Norm}_{L/k}(\mathrm{disc}(\mathcal{Q})) \in k^{\times 2}$, we have that $\mathcal{Q}$ fails to have smooth local points at an* even *number of places of L.*

When $n = 4$, there are exactly five rank 4 quadrics in the pencil of quadrics containing X, so the second condition in case (2b) holds generically and can be easily checked. In particular, it is satisfied whenever there is no pair of Galois conjugate rank 4 quadrics in the pencil or if X has points everywhere locally. For further details of the cases not covered in case (2b) see Remark 6.2 and Section 7.1. Even assuming standard conjectures, our theorem does not rule out the possibility that there exists a smooth complete intersection of two quadrics in $\mathbb{P}^4_k$ over a global field k which does not possess a point over any quadratic extension. Nevertheless, based on our results and extensive numerical computations we expect that the following question has positive answer.

Question 1.3. *Does every complete intersection of 2 quadrics $X \subset \mathbb{P}^4_k$ over a number field k possess a K-rational point for some quadratic extension K/k?*

1.1. **Obstructions to index 1 over local and global fields.** Over local and global fields, necessary and sufficient conditions for an intersection of two quadrics to have index 1 (equivalently, to have a rational point) have been well studied. When k is a local field and $n \leq 7$ there are examples with $X(k) = \emptyset$ (which necessarily have index greater than 1), while for $n \geq 8$ and k a p-adic field, $X(k) \neq \emptyset$ [Dem56]. For k a number field, recent work of Heath-Brown [HB18], building on [CTSSD87a, CTSSD87b], shows that the Hasse principle holds when $n \geq 7$. Hence for $n \geq 8$ the only obstruction to the existence of a rational point is at the archimedean primes. Heath-Brown's result gives positive evidence for the conjecture of Colliot-Thélène, Sansuc and Swinnerton-Dyer that X satisfies the Hasse principle as soon as $n \geq 5$ [CTSSD87b, §16].

When $n = 4$ (in which case X is a del Pezzo surface of degree 4), the Hasse principle can fail [BSD75]. Colliot-Thélène and Sansuc have conjectured that this failure is always explained by the Brauer-Manin obstruction [CTS80]. In fact, this conjecture for $n = 4$ implies the Hasse principle for $n \geq 5$. Some cases of the $n = 4$ conjecture have been proven conditionally on Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups of elliptic curves by Wittenberg [Wit07], thereby giving a conditional proof of the Hasse principle when $n \geq 5$.

1.2. **Outline of the proof of Theorems 1.1 and 1.2.** Using an argument of Wittenberg [Wit07] (which we review in Section 6.2), we can reduce to the case $n = 4$, when X is a del Pezzo surface of degree 4. In Section 4 we prove that any del Pezzo surface of degree 4 over a local field of characteristic not equal to 2 must have points over some quadratic extension, which proves Theorem 1.2(1). Over a global field, this shows that after base change to a suitable quadratic extension X becomes everywhere locally soluble. While it is also true that the Brauer group of X becomes constant after a suitable quadratic extension, one cannot deduce Theorem 1.2(2) directly from case (1) in this way because, in general, there is no quadratic extension K/k for which X_K is locally soluble and the Brauer group of X_K is trivial modulo constant algebras (See Example 6.4).

To obtain our results when k is a global field we study the arithmetic of the symmetric square of X, which is birational to the variety $\mathcal{G}$ parameterizing lines on the quadrics in the pencil of quadrics in $\mathbb{P}^4_k$ containing X (see Section 3 for more details). In Section 5, we develop the main tools for studying the arithmetic of $\mathcal{G}$ over a global field. We determine explicit central simple algberas over the function field of $\mathcal{G}$ representing the Brauer group of $\mathcal{G}$ modulo constant algebras and then proceed to develop techniques to calculate the evaluation maps of these central simple algebras at several types of local points.

These results are used in Section 6 to show further that there is always an adelic 0-cycle of degree 1 on $\mathcal{G}$ orthogonal to the Brauer group and, under the hypothesis of (2b), that there is an adelic point on $\mathcal{G}$ orthogonal to the Brauer group. This is perhaps surprising given that (even assuming (2b)) the Brauer group of $\mathcal{G}$ can contain nonconstant algebras and in general can obstruct weak approximation on $\mathcal{G}$ (see Corollary 6.3 and Example 6.4).

The variety of lines on a smooth quadric 3-fold is a Severi-Brauer 3-fold, so the arithmetic of $\mathcal{G}$ is amenable to the fibration method, as first observed in [CTS82]. Results of [CTSD94] show that, in the number field case, the vanishing of the Brauer-Manin obstruction on $\mathcal{G}$ implies the existence of a 0-cycle of degree 1 on $\mathcal{G}$ and, conditionally on Schinzel's hypothesis, a k-rational point on $\mathcal{G}$. This yields a 0-cycle of degree 2 on X and a quadratic point on X if we assume Schinzel's hypothesis. One can ask whether $\mathrm{index}(\mathcal{G}) = 1$ always implies that $\mathcal{G}$ has a rational point (when k is a global field this is equivalent to Question 1.3). Our results do not answer this question, but they do show that a stronger condition on 0-cycles does not hold. Namely, $\mathcal{G}$ can contain 0-cycles of degree 1 that are *not* rationally equivalent to a rational point (See Remark 7.4(1)).

To deduce the results in case (2) of Theorem 1.2 assuming that Brauer-Manin is the only obstruction to the Hasse principle for del Pezzo surfaces of degree 4 (without assuming Schinzel), we make use of Proposition 2.6 below, which may be of interest in its own right. It relates the Brauer-Manin obstruction on the symmetric square of a variety to the Brauer-Manin obstruction over quadratic extensions. In a similar spirit, we answer a question posed in [CTP00] concerning Brauer-Manin obstructions over extensions (see Remarks 7.4(2)) and give an example of a del Pezzo surface of degree 4 defined over $\mathbb{Q}$ which, for any finite extension $k/\mathbb{Q}$, has a Brauer-Manin obstruction to the existence of k-points if and only if k is of odd degree over $\mathbb{Q}$ (See Section 7.2).

Notation. For a field k we use $\overline{k}$ to denote a separable closure and use $G_k := \mathrm{Gal}(\overline{k}/k)$ to denote the absolute Galois group of k. For k-schemes $Y \to \mathrm{Spec}(k)$ and $S \to \mathrm{Spec}(k)$ we define $Y_S := Y \times_{\mathrm{Spec}(k)} S$ and $\overline{Y} = Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}(\overline{k})$. When $S = \mathrm{Spec}(A)$ is the spectrum of a k-algebra A, we use the notation $Y_A := Y_{\mathrm{Spec}(A)}$. A quadratic point on Y is a morphism of k-schemes $\mathrm{Spec}(K) \to Y$, where K is an étale k-algebra of degree 2. In particular, $K = k \times k$ is allowed in which case $Z_K \simeq Z \times Z$ for any k-subscheme $Z \subset Y$.

The Brauer group of a scheme Y is the étale cohomology group $\mathrm{Br}(Y) := \mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m)$; when $Y = \mathrm{Spec}(R)$ is the spectrum of a ring R we define $\mathrm{Br}(R) := \mathrm{Br}(\mathrm{Spec}\, R)$. If $s_Y : Y \to \mathrm{Spec}(k)$ is a k-scheme, then $\mathrm{Br}_0(Y) \subset \mathrm{Br}(Y)$ is the image of the pullback map $s_Y^* : \mathrm{Br}(k) \to \mathrm{Br}(Y)$. An element $\beta \in \mathrm{Br}(Y)$ may be evaluated at a k-point $y : \mathrm{Spec}(k) \to Y$ by pulling back along y to obtain $\beta(y) := y^*\beta \in \mathrm{Br}(k)$. For a finite locally free morphism of schemes $Y \to Z$

we use $\mathrm{Cor}_{Y/Z}\colon \mathrm{Br}(Y) \to \mathrm{Br}(Z)$ to denote the corestriction map. When $Y = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ are affine schemes this is also denoted by $\mathrm{Cor}_{A/B}\colon \mathrm{Br}(A) \to \mathrm{Br}(B)$.

If Y is an integral k-scheme, $\mathbf{k}(Y)$ denotes its function field. More generally, if Y is a finite union of integral k-schemes Y_i, then $\mathbf{k}(Y) := \prod \mathbf{k}(Y_i)$ is the ring of global sections of the sheaf of total quotient rings. In particular, if a finite dimensional étale k-algebra A decomposes as a product $A \simeq \prod k_j$ of finite field extensions of k, then $\mathbf{k}(Y_A) \simeq \prod \mathbf{k}(Y_{k_j})$, and $\mathrm{Cor}_{\mathbf{k}(Y_A)/\mathbf{k}(Y)} = \sum \mathrm{Cor}_{\mathbf{k}(Y_{k_j})/\mathbf{k}(Y)}$.

A variety over k is a separated scheme of finite type over k. A variety is called nice if it is smooth, projective and geometrically integral and is called split if it contains an open subscheme that is geometrically integral.

For a global field k, we use Ω_k to denote the set of primes of k. For a prime $v \in \Omega_k$ we use k_v to denote the corresponding completion and for a k-scheme Y we set $Y_v := Y_{k_v}$. We use $\mathbb{A}_k$ to denote the adele ring of k. For a subgroup $B \subset \mathrm{Br}(Y)$, $Y(\mathbb{A}_k)^B \subset Y(\mathbb{A}_k)$ denotes the set of adelic points orthogonal to B, i.e.,

$$Y(\mathbb{A}_k)^B = \{(y_v) \in Y(\mathbb{A}_k)\ :\ \forall\, \beta \in B\,,\ \sum_{v\in\Omega_k} \mathrm{inv}_v(\beta(y_v)) = 0\,\}\,.$$

We define $Y(\mathbb{A}_k)^{\mathrm{Br}} := Y(\mathbb{A}_k)^{\mathrm{Br}(Y)}$.

If Q is a quadratic form on a vector space V over a field F then (by definition) the mapping $B_Q : V \times V \to F$ given by $B_Q(x, y) = Q(x + y) - Q(x) - Q(y)$ is bilinear. We say that Q is regular if the set $\{x \in V\ ;\ \forall\, y \in\ B_Q(x, y) = 0 \text{ and } Q(x) = 0\}$ contains only the zero vector in V. (If the characteristic of F is not 2, then the condition $Q(x) = 0$ is superfluous.) Then Q is regular if and only if the quadric $\mathcal{Q}$ in $\mathbb{P}(V)$ defined by the vanishing of Q is regular (see [EKM08, Proposition 22.1]). We define the rank $r(Q)$ of Q to be the largest integer m such that there is a subspace $W \subset V$ of dimension m such that the restriction of Q to W is geometrically regular, i.e., such that the intersection of $\mathcal{Q}$ with the linear space corresponding to W is smooth. The rank of a quadric in $\mathbb{P}^n$ is defined to be the rank of any quadratic form defining it.

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2. Brauer-Manin obstructions over extensions

In this section, we prove some general results relating the Brauer-Manin obstruction on a nice variety Y to the Brauer-Manin obstruction over an extension. Moreover, for quadratic extensions, we relate the Brauer-Manin obstruction on (a desingularization of) the symmetric square to the Brauer-Manin obstruction over quadratic extensions.

Lemma 2.1. *Let Y/k be a nice variety over a global field k, let K/k be a finite extension, and let B be a subset of $\mathrm{Br}(Y_K)$. Then $Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(B)} \subset Y(\mathbb{A}_K)^B$. In particular,*

(1) if $Y(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $Y_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$, and
(2) for any $d \mid [K:k]$, $Y(\mathbb{A}_k) \subset Y_K(\mathbb{A}_K)^{\mathrm{Res}_{K/k}\mathrm{Br}(Y)[d]}$.

Proof. By [CTS21, Prop. 3.8.1], for any $\alpha \in \mathrm{Br}(Y_K)$ and for any local point $P_v \in Y(k_v)$, we have $(\mathrm{Cor}_{Y_K/Y}(\alpha))(P_v) = \mathrm{Cor}_{K_v/k_v}(\alpha(P_v))$, where $K_v = K \otimes_k k_v$. Thus, for $(P_v) \in Y(\mathbb{A}_k)$,

$$\sum_{v\in\Omega_k} \mathrm{inv}_v\left(\mathrm{Cor}_{Y_K/Y}(\alpha)(P_v)\right) = \sum_{v\in\Omega_k} \mathrm{inv}_v\left(\mathrm{Cor}_{K_v/k_v}(\alpha(P_v))\right) = \sum_{v\in\Omega_k}\sum_{w\in\Omega_K, w|v} \mathrm{inv}_w(\alpha(P_v))$$

(where the last equality follows from the equality of maps $\mathrm{inv}_w = \mathrm{inv}_v \circ \mathrm{Cor}_{K_w/k_v}$ for any place $w|v$), and so $Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(\alpha)} \subset Y(\mathbb{A}_K)^{\alpha}$. The general statement follows by considering the intersection of $Y(\mathbb{A}_K)^{\alpha}$ for all $\alpha \in B$.

It remains to prove statements (1) and (2). The first follows from taking $B = \mathrm{Br}(Y_K)$ and observing that $Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(\mathrm{Br}(Y_K))}$, and the second follows from taking $B = \mathrm{Res}_{K/k}\mathrm{Br}(Y)[d]$ and using that $\mathrm{Cor}_{K/k} \circ \mathrm{Res}_{K/k} = [K:k]$. ◻

Remark 2.2. Yang Cao has given an alternative proof of Lemma 2.1(1) which also yields a similar statement for the étale-Brauer obstruction. This will appear in forthcoming work of Yang Cao and Yongqi Liang.

Lemma 2.3. *Let Y be a nice variety over a global field k. Assume:*

(1) $\mathrm{Pic}(\overline{Y})$ is finitely generated and torsion free,
(2) $\mathrm{Br}(\overline{Y})$ is finite, and
(3) $\mathrm{Br}(Y) \to \mathrm{Br}(\overline{Y})^{G_k}$ is surjective.

Then there is a finite Galois extension k_1/k such that for all extensions K/k linearly disjoint from k_1 the map $\mathrm{Res}_{K/k}\colon \mathrm{Br}(Y)/\mathrm{Br}_0(Y) \to \mathrm{Br}(Y_K)/\mathrm{Br}_0(Y_K)$ is surjective.

Proof. Assumption (1) implies that $\mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) \simeq \mathrm{H}^1(k_0/k, \mathrm{Pic}(\overline{Y}))$ for some finite Galois extension k_0/k. By assumption (2) there is a finite Galois extension k_1/k_0 such that $\mathrm{Res}_{\overline{k}/k_1}\colon \mathrm{Br}(Y_{k_1}) \to \mathrm{Br}(\overline{Y})$ is surjective. Now suppose K/k is linearly disjoint from k_1. In particular, K is linearly disjoint from k_0, so $\mathrm{Res}_{K/k}\colon \mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) \to \mathrm{H}^1(K, \mathrm{Pic}(\overline{Y})) \simeq \mathrm{Br}_1(Y_K)/\mathrm{Br}_0(Y_K)$ is an isomorphism. So it will suffice to show that $\mathrm{Br}(Y)$ and $\mathrm{Br}(Y_K)$ have the same image in $\mathrm{Br}(\overline{Y})$. Since $\mathrm{Br}(Y_{k_1}) \to \mathrm{Br}(\overline{Y})$ is surjective, the image of $\mathrm{Br}(Y_K) \to \mathrm{Br}(\overline{Y})$ is contained in $\mathrm{Br}(\overline{Y})^{G_K} \cap \mathrm{Br}(\overline{Y})^{G_{k_1}}$, which is equal to $\mathrm{Br}(\overline{Y})^{G_k}$, since k_1 and K are linearly disjoint. Thus, by assumption (3), $\mathrm{Br}(Y)$ and $\mathrm{Br}(Y_K)$ have the same image in $\mathrm{Br}(\overline{Y})$. ◻

From Lemmas 2.1(2) and 2.3, we can immediately deduce the following corollary.

Corollary 2.4. *If Y is locally soluble and $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is generated by the image of $\operatorname{Br}(Y)[d]$, then for any extension K/k of degree d, $Y_K(\mathbb{A}_K)^{\operatorname{Res}_{K/k}(\operatorname{Br}(Y))} \neq \emptyset$. Moreover, if Y satisfies the conditions of Lemma 2.3, then there is a finite extension k_1/k such that for any degree d extension K/k which is linearly disjoint from k_1 we have $Y_K(\mathbb{A}_K)^{\operatorname{Br}} \neq \emptyset$.* □

Remark 2.5. If Y is a locally soluble del Pezzo surface of degree 4 the corollary applies with $d = 2$. This gives a proof of the $n = 4$ case of Theorem 1.2(2) under the additional hypothesis of local solubility. Note that local solubility is used here in two distinct ways. First it ensures that $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is generated by the image of $\operatorname{Br}(Y)[2]$ (which is not the case in general even though $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is 2-torsion). Second, it implies that the canonical maps $\operatorname{Br}(k) \to \operatorname{Br}_0(Y)$ are isomorphisms, locally and globally. This is used implicitly in the proof of Lemma 2.1. In general, $\operatorname{Br}(k) \to \operatorname{Br}_0(Y)$ need not be injective (see Lemma 5.8 for a description of the kernel when Y is a del Pezzo surface of degree 4) and so $\operatorname{Res}_{K/k}$ does not necessarily annihilate $[K : k]$-torsion elements of $\operatorname{Br}_0(Y)$. Consequently, the exact sequence

$$0 \to \operatorname{Br}(k) \to \bigoplus \operatorname{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

of global class field theory has no analogue for $\operatorname{Br}_0(Y)$.

The following proposition relates the Brauer-Manin obstruction over quadratic extensions to the Brauer-Manin obstruction on the symmetric square. Note that while the symmetric square is singular if Y has dimension at least 2, there exist smooth projective models over any field, e.g., $\operatorname{Hilb}^2(Y)$ [Che98, Theorem 3.0.1 and equation (0.2.1)].

Proposition 2.6. *Let Y be a nice variety of dimension at least 2 over a field k and let $Y^{(2)}$ be a smooth projective model of the symmetric square of Y over k.*

(1) There is an injective map

$$\operatorname{Cor} \circ \pi_1^* \colon \frac{\operatorname{Br}(Y)}{\operatorname{Br}_0(Y)} \hookrightarrow \frac{\operatorname{Br}(Y^{(2)})}{\operatorname{Br}_0(Y^{(2)})},$$

where the map $\pi_1^* \colon \operatorname{Br}(Y) \to \operatorname{Br}(Y^2)$ is induced by projection onto the first factor and the map $\operatorname{Cor} \colon \operatorname{Br}(Y^2) \to \operatorname{Br}(Y^{(2)})$ is the corestriction map corresponding to the field extension $\mathbf{k}(Y^{(2)}) = \mathbf{k}(\operatorname{Sym}^2(Y)) \hookrightarrow \mathbf{k}(Y^2)$ induced by the canonical rational map $Y^2 \dashrightarrow \operatorname{Sym}^2(Y)$.

(2) Let $\alpha \in \operatorname{Br}(Y)$ and $\beta = \operatorname{Cor} \circ \pi_1^*(\alpha) \in \operatorname{Br}(Y^{(2)})$. For any $y \in Y^{(2)}$ that corresponds to a quadratic point $\tilde{y} \colon \operatorname{Spec}(K) \to Y$ for some étale $\mathbf{k}(y)$-algebra K of degree 2 over $\mathbf{k}(y)$, we have $\beta(y) = \operatorname{Cor}_{K/\mathbf{k}(y)}(\alpha(\tilde{y}))$.

(3) Suppose k is a global field of characteristic not equal to 2 and let $B \subset \operatorname{Br}(Y^{(2)})/\operatorname{Br}_0(Y^{(2)})$ denote the image of the map in (1). If there exists a quadratic extension K/k such that $Y_K(\mathbb{A}_K)^{\operatorname{Br}(Y_K)} \neq \emptyset$, then $Y^{(2)}(\mathbb{A}_k)^B \neq \emptyset$.

(4) Suppose k is a global field of characteristic not equal to 2. Let $B \subset \operatorname{Br}(Y^{(2)})/\operatorname{Br}_0(Y^{(2)})$ denote the image of the map in (1). Suppose that $Y^{(2)}(\mathbb{A}_k)^B \neq \emptyset$ and that Y satisfies the hypotheses of Lemma 2.3. Then there exists a finite set $S \subset \Omega_k$, degree 2 étale k_v-algebras K_w/k_v for $v \in S$ and a finite extension k_1/k such that for any quadratic extension K/k that is linearly disjoint from k_1 and such that $K \otimes k_v \simeq K_w$ for $v \in S$ we have $Y_K(\mathbb{A}_K)^{\operatorname{Br}} \neq \emptyset$. In particular, there are infinitely many quadratic extensions K/k such that $Y_K(\mathbb{A}_K)^{\operatorname{Br}} \neq \emptyset$.

Proof. (1)**:** Let $U \subset Y^2$ be the complement of the diagonal in Y^2. The restriction of $f\colon Y^2 \dashrightarrow \mathrm{Sym}^2(Y)$ to U gives a dominant morphism $f\colon U \to \mathrm{Sym}^2(Y)$ whose image is the regular locus $\mathrm{Sym}^2(Y)_{\mathrm{reg}}$ of $\mathrm{Sym}^2(Y)$. Since $\mathbf{k}(Y^2) = \mathbf{k}(U)$ is Galois over $\mathbf{k}(\mathrm{Sym}^2(Y)_{\mathrm{reg}})$ with Galois group generated by the involution σ interchanging the factors of $Y \times Y$, by [GS06, Chapter 3, Exercise 3], the composition

$$\mathrm{Res}_{U/\mathrm{Sym}^2(Y)_{\mathrm{reg}}} \circ \mathrm{Cor}_{U/\mathrm{Sym}^2(Y)_{\mathrm{reg}}}\colon \mathrm{Br}(\mathbf{k}(U)) \to \mathrm{Br}\left(\mathbf{k}(\mathrm{Sym}^2(Y)_{\mathrm{reg}})\right)$$

is given by $x \mapsto x + \sigma(x)$. We may then deduce that the same formula holds for the composition $\mathrm{Res}_{U/\mathrm{Sym}^2(Y)_{\mathrm{reg}}} \circ \mathrm{Cor}_{U/\mathrm{Sym}^2(Y)_{\mathrm{reg}}}\colon \mathrm{Br}(U) \to \mathrm{Br}(\mathrm{Sym}^2(Y)_{\mathrm{reg}})$ by evaluating at generic points.

Note that $Y^2 - U$ is isomorphic to Y, and so has codimension $\dim(Y) \geq 2$ in Y^2. Therefore, $\mathrm{Br}(Y^2) = \mathrm{Br}(U)$ as subgroups of $\mathrm{Br}(\mathbf{k}(Y^2))$. Similarly, we obtain $\mathrm{Br}(Y^{(2)}) = \mathrm{Br}(\mathrm{Sym}^2(Y)_{\mathrm{reg}})$. So, we have a map $\mathrm{Cor} \circ \pi_1^*\colon \mathrm{Br}(Y) \to \mathrm{Br}(Y^{(2)})$ which, when composed with $\mathrm{Res}\colon \mathrm{Br}(Y^{(2)}) = \mathrm{Br}(\mathrm{Sym}^2(Y)_{\mathrm{reg}}) \to \mathrm{Br}(U) = \mathrm{Br}(Y^2)$ is equal to the diagonal map $\mathrm{Br}(Y) \to \mathrm{Br}(Y) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(Y^2)$ sending α to $\pi_1^*\alpha + \sigma(\pi_1^*\alpha) = \pi_1^*\alpha + \pi_2^*\alpha$. The kernel of this is equal to $\mathrm{Br}_0(Y)[2]$, so $\mathrm{Res} \circ \mathrm{Cor} \circ \pi_1^*$ and, consequently, $\mathrm{Cor} \circ \pi_1^*$ induce injective maps on $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$.

(2)**:** The points y and $\tilde{y}$ fit into a commutative diagram displayed on the left below. This induces the diagram displayed on the right. Commutativity of the later gives the result.

$$\begin{tikzcd}
\mathrm{Spec}(\mathbf{k}(y)) \arrow[r, "y"] & Y^{(2)} \\
f^{-1}(y) \arrow[u] \arrow[r] \arrow[d, equal] & Y \times Y \arrow[u, dotted, "f"] \arrow[d, "\pi_1"] \\
\mathrm{Spec}(K) \arrow[r, "\tilde{y}"] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathrm{Br}(\mathbf{k}(y)) & \mathrm{Br}(Y^{(2)}) \arrow[l, "y^*"'] \\
\mathrm{Br}(f^{-1}(y)) \arrow[u, "\mathrm{Cor}"] \arrow[d, equal] & \mathrm{Br}(Y \times Y) \arrow[l] \arrow[u, "\mathrm{Cor}"'] \\
\mathrm{Br}(K) & \mathrm{Br}(Y) \arrow[l, "\tilde{y}^*"] \arrow[u, "\pi_1^*"']
\end{tikzcd}$$

(3)**:** Suppose that K/k is a quadratic extension, $(x_w)_{w \in \Omega_K} \in Y_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$ and that $\beta = \mathrm{Cor}(\pi_1^*\alpha) \in \mathrm{Br}(Y^{(2)})$ represents a class in B that is the image of $\alpha \in \mathrm{Br}(Y)$. For $v \in \Omega_k$ that split in K, let $y_v \in Y^{(2)}(k_v)$ be the point corresponding to the pair $\{x_w : w \mid v\}$. For $v \in \Omega_k$ with a unique $w \in \Omega_K$ dividing v, let $y_v \in Y^{(2)}(k_v)$ be the point corresponding to x_w and its K_w/k_v-conjugate. Note that in each case we may perturb the x_w if needed to ensure we are considering y_v corresponding to a distinct pair of points on Y. This determines an adelic point $y = (y_v) \in Y^{(2)}(\mathbb{A}_k)$. For any $v \in \Omega_k$, (2) gives $\beta(y_v) = \sum_{w|v} \mathrm{Cor}_{K_w/k_v}(\alpha(x_w))$ and consequently, $\mathrm{inv}_v(\beta(y_v)) = \sum_{w|v} \mathrm{inv}_w(\alpha(x_w))$. Since $(x_w)_{w \in \Omega_K} \in Y_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$ we see that $y \in Y^{(2)}(\mathbb{A}_k)^B \neq \emptyset$.

(4)**:** Suppose $(y_v)_{v \in \Omega_k} \in Y^{(2)}(\mathbb{A}_k)^B \neq \emptyset$. Perturbing y_v if needed we may assume that y_v lies in $\mathrm{Sym}^2(Y)_{\mathrm{reg}}$ and hence corresponds to a quadratic point $\tilde{y}_v\colon \mathrm{Spec}(K_w) \to Y$, where K_w is an étale k_v-algebra of degree 2. Moreover, by (2) if $\alpha \in \mathrm{Br}(Y)$ and $\beta = \mathrm{Cor}(\pi_1^*(\alpha))$, then $\beta(y_v) = \mathrm{Cor}_{K_w/k_v}(\alpha(\tilde{y}_v))$. By assumption $\sum_{v \in \Omega_k} \mathrm{inv}_v \beta(y_v) = 0$, so $(\tilde{y}_v)_{v \in \Omega_k}$ is an effective adelic 0-cycle of degree 2 on Y which is orthogonal to the Brauer group of Y. Under the additional hypotheses of (4), $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is finite and, by Lemma 2.3, there is an extension k_1/k such that for K/k linearly disjoint from k_1, $\mathrm{Res}_{K/k}\colon \mathrm{Br}(Y) \to \mathrm{Br}(Y_K)$ is surjective. Moreover, for any set $\alpha_1, \ldots, \alpha_n \in \mathrm{Br}(Y)$ of representatives for $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$, there is a finite set $S \subset \Omega_k$ such that for all $i = 1, \dots, n$ and all $v \notin S$ the evaluation maps $\mathrm{inv}_v \circ \alpha_i \colon Y(k_v) \to \mathbb{Q}/\mathbb{Z}$ are constant (see [CTS13, Lemma 1.2 & Theorem 3.1]). Let K/k be a quadratic extension linearly disjoint from k_1 and such that $K \otimes k_v \simeq K_w$ for $v \in S$. Then any any adelic point $(x_w)_{w\in\Omega} \in Y_K(\mathbb{A}_K)$ such that $\tilde{y}_v = \sum_{w|v} x_w$ for $v \in S$ will be orthogonal to $\mathrm{Br}(Y_K)$. By the Grunwald-Wang theorem the map $k^\times/k^{\times 2} \to \prod_{v\in S} k_v^\times/k_v^{\times 2}$ is surjective, so such extensions K/k do in fact exist. □

3. Pencils of quadrics in $\mathbb{P}^4$ and associated objects

Let $\mathcal{Q} \subset \mathbb{P}^4 \times \mathbb{P}^1$ be a pencil of quadrics over a field k of characteristic different from 2. It will often be convenient to work with a bihomogeneous polynomial Q of degree $(2,1)$ whose vanishing defines $\mathcal{Q}$. If the projection map $\mathcal{Q} \to \mathbb{P}^1$ is generically smooth, then we may naturally associate three objects. First, we may consider the base locus $X = X_{\mathcal{Q}} \subset \mathbb{P}^4$ of the pencil of quadrics, i.e., $\cap_{t\in\mathbb{P}^1} \mathcal{Q}_t$, where $\mathcal{Q}_t \subset \mathbb{P}^4$ denotes the fiber over $t \in \mathbb{P}^1$. This is a degree 4 projective surface. Second, we may consider the subscheme $\mathcal{S} \subset \mathbb{P}^1$ parameterizing the singular quadrics in the pencil, which is given by the vanishing of $\det(M_Q)$, where M_Q denotes the symmetric matrix corresponding to Q considered as a quadratic form whose coefficients are linear polynomials in the homogeneous coordinate ring of $\mathbb{P}^1$. Since $\mathcal{Q} \to \mathbb{P}^1$ is generically smooth, $\mathcal{S} \subset \mathbb{P}^1$ is a degree 5 subscheme. Third, we may consider the fourfold $\mathcal{G} = \mathcal{G}_{\mathcal{Q}} \to \mathbb{P}^1$ that parametrizes lines on quadrics in the pencil; the generic fiber of $\mathcal{G}$ is a Severi-Brauer variety with index dividing 4 and order dividing 2 [EKM08, Ex. 85.4].

Each of these objects has been well-studied, and their conditions for smoothness are known to be closely related.

Proposition 3.1. *Let $\mathcal{Q} \subset \mathbb{P}^4 \times \mathbb{P}^1$ be a pencil of quadrics. Then the following are equivalent:*

1. *The base locus X is smooth and purely of dimension 2, in which case X is a del Pezzo surface of degree 4;*
2. *The degree 5 subscheme $\mathcal{S} \subset \mathbb{P}^1$ is reduced;*
3. *For every $s \in \mathcal{S}$, the fiber $\mathcal{Q}_s$ is rank 4 and the vertex of $\mathcal{Q}_s$ does not lie on any other quadric in the pencil; and*
4. *The fourfold $\mathcal{G}$ is smooth, the map $\mathcal{G} \to \mathbb{P}^1$ is smooth away from $\mathcal{S}$, and above $\mathcal{S}$ the fibers are geometrically reducible.*

Proof. The equivalence of conditions (1), (2), and (3) is given by [Rei72, Prop. 2.1]. The equivalence of (4) with any (equivalently all) of the others is given by [Rei72, Thm. 1.10]. □

Definition 3.2. *A pencil of quadrics $\mathcal{Q}$ satisfies (†) if any of the above conditions hold. Given a pencil $\mathcal{Q}$ satisfying (†), we define $\varepsilon_{\mathcal{S}} \in \mathbf{k}(\mathcal{S})/\mathbf{k}(\mathcal{S})^{\times 2}$ to be the discriminant of a smooth hyperplane section of $\mathcal{Q}_{\mathcal{S}}$; note that the square class of the discriminant does not depend on the choice of hyperplane, nor on the choice of a defining equation for $\mathcal{Q}_{\mathcal{S}}$.*

Given a pencil of quadrics satisfying (†), there are even stronger connections among these three objects.

Proposition 3.3. *Let $\mathcal{Q}$ be a pencil of quadrics satisfying (†). Let $X = X_{\mathcal{Q}}, \mathcal{G} = \mathcal{G}_{\mathcal{Q}}$, and $(\mathcal{S}, \varepsilon_{\mathcal{S}}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$.*

1. *The variety $\mathcal{G}$ is birational to the symmetric square $\mathrm{Sym}^2(X)$ of X. Hence, $\mathcal{G}(k) \neq \emptyset$ if and only if $X(K) \neq \emptyset$ for some quadratic extension K/k.*

(2) The residues of the Brauer class $[\mathcal{G}_{\mathbf{k}(\mathbb{P}^1)}] \in \operatorname{Br} \mathbf{k}(\mathbb{P}^1)$ *are*

$$\varepsilon_{\mathcal{S}} \in \mathbf{k}(\mathcal{S})^{\times}/\mathbf{k}(\mathcal{S})^{\times 2} \simeq \mathrm{H}^1\left(\mathbf{k}(\mathcal{S}), \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \subset \bigoplus_{t \in (\mathbb{P}^1)^{(1)}} \mathrm{H}^1(\mathbf{k}(t), \mathbb{Q}/\mathbb{Z}).$$

In particular, $\operatorname{Norm}_{\mathbf{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{S}}) \in k^{\times 2}$.

(3) Given a pair $(\mathcal{S}', \varepsilon_{\mathcal{S}'})$ *where* $\mathcal{S}' \subset \mathbb{P}^1$ *is a reduced degree 5 subscheme and a class* $\varepsilon_{\mathcal{S}'} \in \mathbf{k}(\mathcal{S}')^{\times}/\mathbf{k}(\mathcal{S}')^{\times 2}$ *of square norm, there exists a unique (up to isomorphism) pencil of quadrics* $\mathcal{Q}$ *such that* $(\mathcal{S}', \varepsilon_{\mathcal{S}'}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$*. Thus, for any* $t \in \mathbb{P}^1 - \mathcal{S}$, $[\mathcal{G}_t] \in \operatorname{Br}(\mathbf{k}(t))$ *is determined by* $(\mathcal{S}, \varepsilon_{\mathcal{S}})$*.*

Remark 3.4. The second statement of Part (2) provides an alternate proof of a proposition by Wittenberg [Wit07, Prop. 3.39].

Proof. (1)**:** Consider a point $(x, x') \in X \times X - \Delta$, where Δ denotes the diagonal image of X, and let $\ell_{\{x,x'\}}$ be the line joining them. Since $\ell_{\{x,x'\}} \cap X$ contains the two points x, x', the line $\ell_{\{x,x'\}}$ must lie on a quadric in the pencil containing X. If $\ell_{\{x,x'\}}$ is not contained in X, then this quadric must be unique. Indeed, the line is contained in $\mathcal{Q}_t$ for $t = t_{\{x,x'\}} := [B_0(x,x') : -B_\infty(x,x')] \in \mathbb{P}^1_k(\overline{k})$, where B_s denotes the symmetric bilinear form associated to Q_s. This gives a dominant rational map

$$f \colon X \times X \dashrightarrow \mathcal{G}, \quad (x, x') \mapsto (t_{\{x,x'\}}, \ell_{\{x,x'\}}),$$

that is generically of degree 2 and factors through the symmetric square of X. Thus, the induced map $\operatorname{Sym}^2 X \dashrightarrow \mathcal{G}$ is birational.

If $\mathcal{G}(k) \neq \emptyset$, then the Lang-Nishimura Theorem (see, e.g., [Poo17, Theorem 3.6.11]) (which applies since $\mathcal{G}$ is smooth) implies that $\operatorname{Sym}^2(X)(k) \neq \emptyset$ and, consequently, that there is a quadratic point on X. In particular, there is a quadratic extension K/k with $X(K) \neq \emptyset$. Conversely, if $X(K) \neq \emptyset$ for some quadratic extension K/k, then $X(K)$ is infinite by [SS91, Theorem (0.1)]. The line through any Galois stable pair of distinct points gives a k-rational point on $\mathcal{G}$.

(2)**:** Let $t \in \mathbb{P}^1$. By [Rei72, Thms. 1.2 and 1.8], the fiber $\mathcal{G}_t$ is smooth and geometrically irreducible exactly when $\mathcal{Q}_t$ has rank 5. Thus, for all $t \in \mathbb{P}^1 - \mathcal{S}$, the class $[\mathcal{G}_{\mathbf{k}(\mathbb{P}^1)}]$ has trivial residue at t. By Proposition 3.1 and assumption (†), if $t \in \mathcal{S}$, then $\mathcal{Q}_t$ has rank 4. If $\mathcal{Q}_t$ is rank 4 and has square discriminant, then by [Rei72, Thm. 1.8] the fiber $\mathcal{G}_t$ is reducible and split over $\mathbf{k}(t)$. If $\mathcal{Q}_t$ is rank 4 and has nonsquare discriminant, then the same result of Reid says that $\mathcal{G}_t$ is irreducible and non-split over $\mathbf{k}(t)$, but becomes split over the quadratic discriminant extension. Thus, the residue of $[\mathcal{G}_{\mathbf{k}(\mathbb{P}^1)}]$ at t is the discriminant of $\mathcal{Q}_t$. By definition of $\varepsilon_{\mathcal{S}}$, this gives the first statement. The second statement now follows from the Faddeev exact sequence for $\operatorname{Br} \mathbf{k}(\mathbb{P}^1)$ (see [GS06, Thm 6.4.5] or (5.4)).

(3)**:** The first statement is a theorem of Flynn [Fly09] which was expanded upon by Skorobogatov [Sko10]. The second statement follows from the first together with the Faddeev exact sequence for $\operatorname{Br}(\mathbf{k}(\mathbb{P}^1))$ (see [GS06, Thm 6.4.5] or (5.4)). □

3.1. **Notation.** Throughout the paper, we will consider only pencils of quadrics that satisfy (†) and we will move freely between the objects $\mathcal{Q}, X = X_{\mathcal{Q}}, \mathcal{G} = \mathcal{G}_{\mathcal{Q}}$, and $(\mathcal{S}, \varepsilon_{\mathcal{S}}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$. We will assume that $\mathcal{S} \subset \mathbb{A}^1 = \mathbb{P}^1 - \infty$. This can be arranged by an automorphism of $\mathbb{P}^1$, provided k has at least 5 elements. We will write $k[T]$ for the coordinate ring of $\mathbb{A}^1$ and let $f(T)$ be the unique monic polynomial whose vanishing defines $\mathcal{S}$.

Let $Q_{\mathbb{A}^1} \in k(T)[x_0, \dots, x_4]$ be a quadratic form whose coefficients are linear polynomials in $k[T]$ and whose vanishing defines $\mathcal{Q}_{\mathbb{A}^1}$ on $\mathbb{A}^1 \subset \mathbb{P}^1$. While $Q_{\mathbb{A}^1}$ is only defined up to multiplication by an element of $k^\times$, none of our results depend on this choice. For a (possibly reducible) subscheme $\mathcal{T} \subset \mathbb{A}^1 = \operatorname{Spec}(k[T])$, the canonical map $k[T] \to \mathbf{k}(\mathcal{T})$ can be applied to the coefficients of $Q_{\mathbb{A}^1}$ to obtain a quadratic form $Q_{\mathcal{T}}$ over the k-algebra $\mathbf{k}(\mathcal{T})$ whose vanishing defines $\mathcal{Q}_{\mathcal{T}} = \mathcal{Q} \times_{\mathbb{P}^1} \mathcal{T}$. In particular, for $a \in k = \mathbb{A}^1(k)$, the form Q_a is obtained by evaluating the coefficients of $Q_{\mathbb{A}^1}$ at a. We define $Q_\infty = Q_1 - Q_0$, so that $Q_{\mathbb{A}^1} = Q_0 + TQ_\infty$.

We will write θ for the image of T in $\mathbf{k}(\mathcal{S}) = k[T]/\langle f(T) \rangle$. For a subscheme $\mathcal{T} \subset \mathcal{S}$ we use $\varepsilon_{\mathcal{T}} \in \frac{\mathbf{k}(\mathcal{T})^\times}{\mathbf{k}(\mathcal{T})^{\times 2}} \subset \frac{\mathbf{k}(\mathcal{S})^\times}{\mathbf{k}(\mathcal{S})^{\times 2}}$ to denote the discriminant corresponding to $\mathcal{Q}_{\mathcal{T}}$. We will use N to denote any map induced in an obvious way by the norm map $\operatorname{Norm}_{\mathbf{k}(\mathcal{S})/k} \colon \mathbf{k}(\mathcal{S}) \to k$. Note that $\operatorname{Norm}_{\mathbf{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}) = \operatorname{Norm}_{\mathbf{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{T}}) = \mathrm{N}(\varepsilon_{\mathcal{T}})$.

4. Quadratic points on del Pezzo surfaces of degree 4 over local fields

In this section we prove the following theorem.

Theorem 4.1. *Let X/k be a del Pezzo surface of degree 4 over a local field of characteristic not equal to 2. There is a quadratic extension K/k such that $X(K) \neq \emptyset$.*

Applying Proposition 3.3(1) yields the following.

Corollary 4.2. *Assume k is a local field of characteristic not equal to 2. For any pencil of quadric threefolds $\mathcal{Q} \to \mathbb{P}^1$ satisfying (†), $\mathcal{G}_{\mathcal{Q}}(k) \neq \emptyset$.* □

The proof of Theorem 4.1 will require the following lemma.

Lemma 4.3. *Suppose Q and $\tilde{Q}$ are quadratic forms of rank $r(Q)$ and $r(\tilde{Q})$, respectively, over a field F. Then $r(Q \perp \tilde{Q}) = r(Q) + r(\tilde{Q})$ except when $\operatorname{char}(F) = 2$ and $r(Q)$ and $r(\tilde{Q})$ are both odd, in which case $r(Q \perp \tilde{Q}) = r(Q) + r(\tilde{Q}) - 1$.*

Proof. For $\operatorname{char}(F) \neq 2$ see [EKM08, Lemma 7.20]. For $\operatorname{char}(F) = 2$ this follows from [EKM08, Corollary 7.31 and Lemma 7.21] and the fact that an orthogonal direct sum of rank 1 forms has rank 1 (cf. [EKM08, Remark 7.24]). □

Proof of Theorem 4.1. If k is archimedean, then $[\overline{k} : k] \leq 2$ so the result is immediate. Henceforth we assume that k is nonarchimedean, and we write $\mathcal{O}$ for the valuation ring of k and $\mathbb{F}$ for the residue field of k. By [Tia17, Theorem 2.7], there is a linear change of coordinates on $\mathbb{P}^4_k$ such that the integral model $\mathcal{X} \subset \mathbb{P}^4_{\mathcal{O}}$ of X is semistable. In particular, by [Tia17, Lemma 2.22(4)], the special fiber of $\mathcal{X}$ is reduced.

If the special fiber is split, i.e., it contains an open subscheme that is geometrically integral, then the special fiber has a smooth $\mathbb{F}'$-point for a sufficiently large extension $\mathbb{F}'/\mathbb{F}$ of *odd* degree. Thus, by Hensel's Lemma, X has a k'-point for k'/k an unramified extension of odd degree, which by the theorems of Amer, Brumer and Springer [Ame76, Bru78, Spr56] show that $X(k) \neq \emptyset$. More generally, the same argument shows that $X(k) \neq \emptyset$ if the special fiber is split over an odd degree extension $\mathbb{F}'/\mathbb{F}$.

Now assume that the special fiber is not split over any odd degree extension. Since an intersection of 2 quadrics has at most 4 irreducible components the Galois action on the geometrically irreducible components must factor through a group of order 2 or 4. If the Galois action factors through a group of order 2, then over a sufficiently large extension $\mathbb{F}'/\mathbb{F}$

of degree congruent to 2 modulo 4, there will be a smooth $\mathbb{F}'/\mathbb{F}$ point. Arguing as in the previous case, we obtain a point on X over the unique unramified quadratic extension k'/k.

It remains to consider the case when the special fiber has four geometrically irreducible components, i.e., when it is geometrically the union of four planes, permuted cyclically by the absolute Galois group of $\mathbb{F}$. By Lemma 4.4 below, then the special fiber is a cone over an intersection of quadrics in $\mathbb{P}^2$ or is geometrically isomorphic to $V(x_0x_1, x_2x_3)$.

Let us first assume that the special fiber is geometrically isomorphic to $V(x_0x_1, x_2x_3)$. Another way to characterize this configuration of four planes is that the intersection of quadrics is a cone over an intersection of quadrics in $\mathbb{P}^3$, and that the pencil of quadrics in $\mathbb{P}^3$ contains exactly two degenerate quadrics, each of which have rank 2. Since the planes are cyclically permuted by Galois, the two rank 2 quadrics in the pencil must also be permuted by Galois. In particular, any $\mathbb{F}$-rational member of the pencil has rank 4. This characterization implies that over $\mathcal{O}$, $\mathcal{X}$ must be given by equations of the form

$$Q(x_0, x_1, x_2, x_3) + \pi^r x_4 \ell(x_0, x_1, x_2, x_3), \text{ and } \tilde{Q}(x_0, x_1, x_2, x_3) + \pi^m x_4^2 + \pi^n x_4 \widetilde{\ell}(x_0, x_1, x_2, x_3),$$

for positive integers r, m, n and any choice of uniformizer π. Since we have assumed that $\mathcal{X}$ is semistable (see [Tia17, Section 2]), the stability condition for the weight vector $(1, 1, 1, 1, 0)$ implies that $m = 1$.

Consider a ramified quadratic extension k'/k and let ϖ is a uniformizer of k'. Over k' we may absorb a ϖ into x_4 and obtain the model $\mathcal{X}'/\mathcal{O}'$ (where $\mathcal{O}'$ is the valuation ring of k'):

$$Q(x_0, \ldots, x_3) + u^r \varpi^{2r-1} x_4 \ell(x_0, \ldots, x_3), \text{ and } \tilde{Q}(x_0, \ldots, x_3) + u x_4^2 + u^n \varpi^{2n-1} x_4 \widetilde{\ell}(x_0, \ldots, x_3),$$

where u is the unit such that $u\varpi^2 = \pi$. Since $Q, \tilde{Q}$ are rank 4 modulo ϖ, Lemma 4.3 gives that $\tilde{Q} + u_1 x_4^2$ is rank 5 modulo ϖ, and every geometric member of the pencil modulo ϖ must have rank at least 3. Thus, by [HB18, Lemma 3.2], the special fiber of $\mathcal{X}'$ is split, so, by the same argument above, $\mathcal{X}'$ has a k'-point.

Now assume that the special fiber is a cone over a reduced intersection of quadrics in $\mathbb{P}^2$. Then, up to a change of variables, $\mathcal{X}$ must be given by quadratic forms of the form

$$g(x_0, x_1, x_2) + \pi^m h(x_3, x_4) + \pi^a x_3 \ell_3(x_0, x_1, x_2) + \pi^b x_4 \ell_4(x_0, x_1, x_2), \text{ and}$$
$$\tilde{g}(x_0, x_1, x_2) + \pi^{\tilde{m}} \tilde{h}(x_3, x_4) + \pi^c x_3 \widetilde{\ell}_3(x_0, x_1, x_2) + \pi^d x_4 \widetilde{\ell}_4(x_0, x_1, x_2),$$

where a, b, c, d, m, and $\tilde{m}$ are positive integers. Since $\mathcal{X}$ is semistable for the weight vector $(1, 1, 1, 0, 0)$, m and $\tilde{m}$ must be equal to 1. Furthermore, since $\mathcal{X}$ is semistable for the weight vector $(1, 1, 1, 1, 0)$, h and $\tilde{h}$ must be relatively prime.

Over a ramified quadratic extension k'/k with uniformizer ϖ, we may absorb a ϖ into x_3 and x_4 and obtain the model $\mathcal{X}'/\mathcal{O}$ given by

$$g(x_0, x_1, x_2) + u h(x_3, x_4) + u^a \varpi^{2a-1} x_3 \ell_3(x_0, x_1, x_2) + u^b \varpi^{2b-1} x_4 \ell_4(x_0, x_1, x_2), \text{ and}$$
$$\tilde{g}(x_0, x_1, x_2) + u \tilde{h}(x_3, x_4) + u^c \varpi^{2c-1} x_3 \widetilde{\ell}_3(x_0, x_1, x_2) + u^d \varpi^{2d-1} x_4 \widetilde{\ell}_4(x_0, x_1, x_2),$$

where u is the unit such that $u\varpi^2 = \pi$. Every member of the pencil modulo ϖ is an orthogonal direct sum of a linear combination of g and $\tilde{g}$ with a linear combination of h and $\tilde{h}$. Since every linear combination of g and $\tilde{g}$ has rank at least 2, Lemma 4.3 implies that every geometric member of the pencil has rank at least 3 modulo ϖ. Furthermore, if the residue characteristic of k' is different from 2 or if either h or $\tilde{h}$ have rank 2, then Lemma 4.3 shows that there is some geometric member of the pencil that has rank at least 5 modulo

ϖ. In these cases [HB18, Lemma 3.2] shows that the special fiber of $\mathfrak{X}'$ is split, so X has a k'-point, by our argument above.

It remains to consider the case when k' has residue characteristic 2 and h and $\tilde{h}$ are both rank 1. After a change of coordinates, the special fiber of $\mathfrak{X}'$ is defined by the vanishing of

$$g(x_0, x_1, x_2) + ux_3^2, \quad \text{and} \quad \tilde{g}(x_0, x_1, x_2) + \tilde{u}x_4^2. \tag{4.1}$$

Since the characteristic is 2, Lemma 4.3 shows that every quadric in this pencil modulo ϖ has rank exactly 3. By [HB18, Proof of Lemma 3.2], the special fiber of $\mathfrak{X}'$ does not contain a quadric of codimension 2. Hence, if the special fiber of $\mathfrak{X}'$ is nonsplit, then it must be the union of 4 planes. By Lemma 4.4, a pencil of quadrics whose base locus is a union of 4 planes must always contain a rank 2 quadric. Since the defining pencil (4.1) of $\mathfrak{X}'$ modulo ϖ does not contain any rank 2 quadrics, the special fiber of $\mathfrak{X}'$ must be split, and hence $X(k') \neq \emptyset$. ☐

Lemma 4.4. *Let $X \subset \mathbb{P}^4$ be a complete intersection of quadrics over an algebraically closed field. If X is the union of 4 distinct planes with a $\mathbb{Z}/4\mathbb{Z}$-action inducing a transitive action on the irreducible components of X, then, up to an automorphism of $\mathbb{P}^4$, X is either a cone over a reduced intersection of quadrics in $\mathbb{P}^2$ or $X = V(x_0x_1, x_2x_3) \subset \mathbb{P}^4$.*

Proof. Let P_1, P_2, P_3, P_4 be the four components of X. First let us consider the case that two of the planes meet only in a point. So without loss of generality, we may assume that $P_1 = V(x_0, x_2)$ and $P_2 = V(x_1, x_3)$. Thus the defining quadrics of X must be contained in $\langle x_0, x_2\rangle\langle x_1, x_3\rangle = \langle x_0x_1, x_1x_2, x_2x_3, x_3x_0\rangle$. Hence, the defining equations for P_3 and P_4 must contain two linearly independent linear combinations of x_0x_1, x_1x_2, x_2x_3, and x_3x_0. In particular, P_3 and P_4 must contain the point of intersection $P_1 \cap P_2$. Thus, X is the cone over an intersection of quadrics in $\mathbb{P}^3$, which is a curve Z of arithmetic genus 1. Since by assumption X is a union of 4 planes, two of which meet only in a point, Z must be the union of 4 lines, two of which must be skew. Thus, by genus considerations together with the $\mathbb{Z}/4\mathbb{Z}$-action, Z must be a 4-gon, i.e., a cycle of rational curves, where each curve meets exactly two of the others.

After a change of coordinates, we may assume that the intersections are

$$P_1 \cap P_3 = V(x_0, x_1, x_2), P_1 \cap P_4 = V(x_0, x_2, x_3), P_2 \cap P_3 = V(x_0, x_1, x_3), P_1 \cap P_4 = V(x_1, x_2, x_3),$$

so $X = V(x_0x_1, x_2x_3)$.

Now assume that all pairs of planes meet in a line. If this line is the same for all pairs, then X is a cone over a reduced intersection of quadrics in $\mathbb{P}^2$. If exactly three of the planes meet in a common line, then without loss of generality the planes must be

$$V(x_0, x_1), V(x_0, x_2), V(x_1, x_2), V(x_3 - \ell(x_0, x_1, x_2), \widetilde{\ell}(x_0, x_1, x_2)).$$

However, there is no intersection of two quadrics that results in these four planes, so this configuration of planes is not possible.

It remains to consider the case when all pairs of planes meet in a line, but no three planes meet in a line. In this case, every triple of planes meets in a point, and these four points must span a 3-dimensional space. Then, after a possible change of coordinates, the planes must be

$$V(x_0, x_1), V(x_0, x_2), V(x_0, x_3), V(x_0, x_4),$$

which again contradicts that X is an intersection of 2 quadrics. ☐

The following provides an alternative proof of the theorem in the case of odd residue characteristic.

Proposition 4.5. *Let $X \subset \mathbb{P}^4_k$ be del Pezzo surface of degree 4 over a local field k of characteristic not equal to 2. Then X has index dividing 2. If the residue characteristic of k is odd, then there is a quadratic extension K/k such that X has a K-point.*

Proof. First let us prove that X has a quadratic point assuming that $s \in \mathcal{S}(k) \neq \emptyset$. After a change of coordinates on the $\mathbb{P}^1$ parametrizing the pencil and a change of coordinates on $\mathbb{P}^4$, we may assume that $s = 0$, that $Q_0 = Q_0(x_0, x_1, x_2, x_3)$, and that $Q_\infty = \tilde{Q}_\infty(x_0, x_1, x_2, x_3) + x_4^2$. If $\mathcal{Q}_0$ contains a smooth k-point, then the line joining the vertex of $\mathcal{Q}_0$ and this point will intersect X in a degree 2 subscheme, which shows that X has a quadratic point. Thus, we may restrict to the case that $\mathcal{Q}_0$ has no smooth k-points.

Projection away from the vertex of $\mathcal{Q}_0 \subset \mathbb{P}^4_k$ gives a double cover $X \to Y := \mathcal{Q}_0 \cap V(x_4)$ onto the quadric surface Y. Since $\mathcal{Q}_0$ has no smooth k-points, $Y(k) = \emptyset$. We will prove that, in this case, the branch curve C of the double cover $X \to Y$ has a quadratic point. Note that by definition of the double cover, $C = X \cap V(x_4)$ and so is a degree 4 genus 1 curve that is the base locus of the pencil of quadric surfaces $\mathcal{Q}' \to \mathbb{P}^1$ with $\mathcal{Q}'_t = \mathcal{Q}_t \cap V(x_4)$. Moreover, C is a 2-covering of the degree 2 genus one curve C' given by the equation $y^2 = \det(M)$ where M is the 4×4 symmetric matrix with entries in $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1))$ corresponding to a defining equation for $\mathcal{Q}'$ (see [AKM+01]).

Consider the fiber of $C' \to \mathbb{P}^1$ above 0. By definition of $\mathcal{Q}'$, this is given by the equation $y^2 = \operatorname{disc}(\mathcal{Q}_0 \cap V(x_4))$. By assumption, $\mathcal{Q}_0 \cap V(x_4)$ has no k-points. Since there is (up to isomorphism) a unique rank 4 quadric over the local field k that is anisotropic and it has square discriminant, we conclude that $\operatorname{disc}(\mathcal{Q}_0 \cap V(x_4))$ is a square and so $C'(k) \neq \emptyset$. Consequently, $C' \simeq \operatorname{Jac}(C)$ and so the order of C in $\mathrm{H}^1(k, \operatorname{Jac}(C))$ divides 2. By a result of Lichtenbaum [Lic69] it follows that C has a point defined over some quadratic extension of the local field k.

Now we can deduce the statement in the proposition. The scheme $\mathcal{S} \subset \mathbb{P}^1_k$ parameterizing singular quadrics in the pencil has degree 5, so there is an odd degree extension k'/k such that $\mathcal{S}(k') \neq \emptyset$. By what we have shown above, X has a K-rational point for some quadratic extension K/k'. It follows that X has index at most 2. If the residue characteristic is odd, then the inclusion $k \subset k'$ induces an isomorphism $k^\times/k^{\times 2} \simeq k'^\times/k'^{\times 2}$, so K contains a quadratic extension k_2/k as an odd index subfield. By the theorems of Amer, Brumer and Springer [Ame76, Bru78, Spr56], we have $X(K) \neq \emptyset \Rightarrow X(k_2) \neq \emptyset$, so X has a k_2-point. □

Remark 4.6. The preceding proof can be adapted to give an easy proof that a locally soluble del Pezzo surface of degree 4 over a global field must have index dividing 2. Indeed, over some odd degree extension X may be written as a double cover of a quadric surface, which is known to satisfy the Hasse principle. Hence X obtains a rational point over some extension of degree $2m$ with m odd.

5. Arithmetic of the space of lines on the quadrics in the pencil

In this section we develop the main tools to prove Theorems 1.1 and 1.2 over global fields. Recall the notation defined in Section 3.1. Throughout, we let $\mathcal{Q} \to \mathbb{P}^1$ be a pencil of quadrics in $\mathbb{P}^4_k$ over an arbitrary field k of characteristic not equal to 2 which satisfies (†), and let $X = X_{\mathcal{Q}}$, $\mathcal{G} = \mathcal{G}_{\mathcal{Q}}$ and $(\varepsilon_{\mathcal{S}}, \mathcal{S}) = (\varepsilon_{\mathcal{S}_{\mathcal{Q}}}, \mathcal{S}_{\mathcal{Q}})$.

In Section 5.1, we compute $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ and construct explicit representatives in $\mathrm{Br}(\mathcal{G})$, denoted by $\beta_{\mathcal{T}}$, which are determined by subsets $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. In Section 5.2, we study the rank 4 quadrics $\mathcal{Q}_{\mathcal{T}}$ corresponding to subsets $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. We use Clifford algebras associated to these rank 4 quadrics to define constant Brauer classes $\mathcal{C}_{\mathcal{T}} \in \mathrm{Br}(k)$ and we show how these are related to the kernel of the canonical map $\mathrm{Br}(k) \to \mathrm{Br}(X)$. The two constructions come together in Sections 5.3 where we show how the $\mathcal{C}_{\mathcal{T}}$ arise when evaluating $\beta_{\mathcal{T}}$ at certain local points of $\mathcal{G}$ (see Lemmas 5.10 and 5.13). Finally, in Section 5.4, we deduce consequences for the evaluation of $\beta_{\mathcal{T}}$ at adelic points of $\mathcal{G}$.

5.1. **The Brauer group of $\mathcal{G}$.** It follows from the Faddeev exact sequence (see [GS06, Thm 6.4.5]) that the homomorphism

$$\gamma' \colon \mathbf{k}(\mathcal{S})^\times \ni \varepsilon \mapsto \mathrm{Cor}_{\mathbf{k}(\mathcal{S})/k}(\varepsilon, T-\theta) \in \mathrm{Br}(\mathbf{k}(\mathbb{P}^1))$$

induces an isomorphism

$$\gamma \colon \ker\left(\mathrm{N} \colon \frac{\mathbf{k}(\mathcal{S})^\times}{\mathbf{k}(\mathcal{S})^{\times 2}} \to \frac{k^\times}{k^{\times 2}}\right) \simeq \ker\left(\mathrm{Br}(\mathbb{P}^1 - \mathcal{S})[2] \xrightarrow{\infty^*} \mathrm{Br}\, k[2]\right), \tag{5.1}$$

where ∞^* denotes evaluation of the Brauer class at $\infty \in \mathbb{P}^1 - \mathcal{S}$. Recall that $\mathrm{N}(\varepsilon_{\mathcal{S}}) \in k^{\times 2}$ by Proposition 3.3(2).

Define $\beta = \pi^*\gamma \colon \ker\left(\mathrm{N} \colon \frac{\mathbf{k}(\mathcal{S})^\times}{\mathbf{k}(\mathcal{S})^{\times 2}} \to \frac{k^\times}{k^{\times 2}}\right) \to \mathrm{Br}(\mathbf{k}(\mathcal{G}))$. For $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, we set $\beta_{\mathcal{T}} := \beta(\varepsilon_{\mathcal{T}})$.

Proposition 5.1. *The map β induces a homomorphism*

$$\ker\left(\mathrm{N} \colon \bigoplus_{s \in \mathcal{S}} \langle \varepsilon_s \rangle \to k^\times/k^{\times 2}\right) \xrightarrow{\beta} \mathrm{Br}(\mathcal{G}), \tag{5.2}$$

whose image generates $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$. Furthermore, $\beta_{\mathcal{S}} = [\mathcal{G}_\infty] \in \mathrm{Br}_0(\mathcal{G})$, and for all $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\varepsilon_{\mathcal{T}} \neq \varepsilon_{\mathcal{S}} \in \mathbf{k}(\mathcal{S})^\times/\mathbf{k}(\mathcal{S})^{\times 2}$, we have

$$\beta_{\mathcal{T}} \in \mathrm{Br}_0(\mathcal{G}) \subset \mathrm{Br}(\mathcal{G}) \iff \beta_{\mathcal{T}} = 0 \in \mathrm{Br}(\mathcal{G}) \iff \varepsilon_{\mathcal{T}} \in \mathbf{k}(\mathcal{T})^{\times 2}.$$

Corollary 5.2.

(1) $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) \simeq (\mathbb{Z}/2\mathbb{Z})^n$ for some $n \in \{0,1,2\}$.

(2) Every nontrivial element of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is represented by $\beta_{\mathcal{T}}$ for some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$.

(3) If $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is not cyclic, then every degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ must *be reducible.*

(4) Let $s_0 \in \mathcal{S}(k)$ be such that there exists an $s' \in \mathcal{S}(k)$ with $\beta_{\{s_0,s'\}} \in \mathrm{Br}(\mathcal{G}) - \mathrm{Br}_0(\mathcal{G})$. Then $\{\beta_{\{s_0,s\}} : s \in \mathcal{S}(k),\ \mathrm{N}(\varepsilon_{\{s_0,s\}}) \in k^{\times 2}\}$ generates $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$.

(5) There is a collection $\mathbb{T}$ of degree 2 subschemes of $\mathcal{S}$ and an element $\varepsilon \in k^\times$, such that
- *$\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ for all $\mathcal{T} \in \mathbb{T}$;*
- *$\{\beta_{\mathcal{T}} : \mathcal{T} \in \mathbb{T}\}$ generates $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$;*
- *for all $s \in \cup_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$, the image of ε in $\mathbf{k}(s)^\times/\mathbf{k}(s)^{\times 2}$ is equal to ε_s; and*
- *for any extension L/k and any $s \in \cup_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$, $\varepsilon \in \mathbf{k}(s_L)^{\times 2}$ if and only if $\varepsilon \in \mathbf{k}(s'_L)^{\times 2}$ for all $s' \in \cup_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$.*

(6) $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$.

(7) *If k is a local or global field, then the injective map* $\operatorname{Br}(X)/\operatorname{Br}_0(X) \to \operatorname{Br}(\mathcal{G})/\operatorname{Br}_0(\mathcal{G})$ *given by Proposition 2.6(1) is an isomorphism.*

Proof of Corollary 5.2. Statements (1)–(4) follow from a straightforward case by case analysis of the possible relations on $\oplus_{s\in\mathcal{S}}\langle\varepsilon_s\rangle$. Given this characterization of $\operatorname{Br}(\mathcal{G})/\operatorname{Br}_0(\mathcal{G})$ in terms of degree 2 subschemes $\mathcal{T}\subset\mathcal{S}$, (5) can be established using [VAV14, Lemma 3.1] for the existence of $\varepsilon\in k^\times$ and (6) follows from [VAV14, Proof of Theorem 3.4]. Finally, when k is a local or global field, the injective map $\operatorname{Br}(X)/\operatorname{Br}_0(X)\to \mathrm{H}^1(k,\operatorname{Pic}(\overline{X}))$ is an isomorphism, so (6) implies that the injective map $\operatorname{Br}(X)/\operatorname{Br}_0(X)\to\operatorname{Br}(\mathcal{G})/\operatorname{Br}_0(\mathcal{G})$ is also surjective. ☐

Remark 5.3. If $\mathcal{T}\subset\mathcal{S}$ is a degree 2 subscheme with $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$ such that the quadric $\mathcal{Q}_{\mathcal{T}}$ has a smooth $\mathbf{k}(\mathcal{T})$-point, then [VAV14, Cor. 3.5] yields a rational map $\rho\colon X \dashrightarrow \mathbb{P}^1$ such that $\rho^*\gamma(\varepsilon_{\mathcal{T}})\in\operatorname{Br}(X)$. One can show that the image of $\rho^*\gamma(\varepsilon_{\mathcal{T}})$ under the map $\operatorname{Br}(X)/\operatorname{Br}_0(X)\to\operatorname{Br}(\mathcal{G})/\operatorname{Br}_0(\mathcal{G})$ given by Proposition 2.6(1) is equal to the class of $\beta_{\mathcal{T}}$.

Proof of Proposition 5.1. Let $\eta\in\mathbb{P}^1$ be the generic point. Since $\mathcal{G}$ is smooth, $\operatorname{Br}(\mathcal{G})$ injects into $\operatorname{Br}(\mathcal{G}_\eta)$. Further, by the Hochschild-Serre spectral sequence, we have an exact sequence

$$0\to\operatorname{Pic}(\mathcal{G}_\eta)\to\left(\operatorname{Pic}(\overline{\mathcal{G}_\eta})\right)^{G_{\mathbf{k}(\eta)}}\to\operatorname{Br}(\mathbf{k}(\eta))\to\ker\left(\operatorname{Br}(\mathcal{G}_\eta)\to\operatorname{Br}(\overline{\mathcal{G}_\eta})\right)\to\mathrm{H}^1\left(G_{\mathbf{k}(\eta)},\operatorname{Pic}(\overline{\mathcal{G}_\eta})\right).$$

Since $\mathcal{G}_\eta$ is a Severi-Brauer variety, $\operatorname{Pic}(\overline{\mathcal{G}_\eta})\simeq\mathbb{Z}$ with trivial Galois action, and $\operatorname{Br}(\overline{\mathcal{G}_\eta})=0$. Hence, the exact sequence simplifies to

$$\mathbb{Z}\to\operatorname{Br}(\mathbf{k}(\eta))\xrightarrow{\pi^*}\operatorname{Br}(\mathcal{G}_\eta)\to 0, \tag{5.3}$$

where the first map sends 1 to $[\mathcal{G}_\eta]\in\operatorname{Br}(\mathbf{k}(\eta))$. Thus, to determine $\operatorname{Br}(\mathcal{G})$, it suffices to determine $\operatorname{Br}(\mathcal{G})\cap\pi^*\operatorname{Br}(\mathbf{k}(\eta))$.

The projection map $\pi\colon\mathcal{G}\to\mathbb{P}^1$ induces the following commutative diagram of exact sequences where the top row is the Faddeev exact sequence [GS06, Thm 6.4.5].

$$\begin{tikzcd}
\operatorname{Br}(k) \arrow[r, hook] \arrow[d, "\pi^*"] & \operatorname{Br}(\mathbf{k}(\eta)) \arrow[r, "(\partial_t)"] \arrow[d, two heads, "(\pi^*)_{\mathrm{Br}}"] & \bigoplus_{t\in(\mathbb{P}^1)^{(1)}}\mathrm{H}^1(\mathbf{k}(t),\mathbb{Q}/\mathbb{Z}) \arrow[r, two heads, "\sum_t \operatorname{Cor}_{\mathbf{k}(t)/k}"] \arrow[d, "(\pi^*)_{\mathrm{H}^1}"] & \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z}) \\
\operatorname{Br}(\mathcal{G}) \arrow[r, hook] & \operatorname{Br}(\mathcal{G}_\eta) \arrow[r, "(\partial_x)"] & \bigoplus_{t\in\mathbb{P}^1}\bigoplus_{\substack{x\in\mathcal{G}^{(1)}\\ \pi(x)=t}}\mathrm{H}^1(\mathbf{k}(x),\mathbb{Q}/\mathbb{Z}), &
\end{tikzcd} \tag{5.4}$$

If $t\in\mathbb{P}^1-\mathcal{S}$, then the fiber $\mathcal{G}_t$ is geometrically irreducible and hence $\pi^*\colon\ \mathrm{H}^1(\mathbf{k}(t),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(\mathbf{k}(\mathcal{G}_t),\mathbb{Q}/\mathbb{Z})$ is an injection. For $t\in\mathcal{S}$, the fiber $\mathcal{G}_t$ consists of two split components that are conjugate over $\mathbf{k}(t)(\sqrt{\varepsilon_t})$.

Therefore, for $t\in\mathcal{S}$, the kernel of $\pi^*\colon\ \mathrm{H}^1(\mathbf{k}(t),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(\mathbf{k}(\mathcal{G}_t),\mathbb{Q}/\mathbb{Z})$ is the 2-torsion cyclic subgroup corresponding to the extension $\overline{k}\cap\mathbf{k}(\mathcal{G}_t)=\mathbf{k}(t)(\sqrt{\varepsilon_t})$. Moreover, the residue of the kernel $(\partial_t\left(\ker(\pi^*)_{\mathrm{Br}}\right))=(\varepsilon_t)_{t\in\mathcal{S}}=\varepsilon_{\mathcal{S}}\in\mathbf{k}(\mathcal{S})/\mathbf{k}(\mathcal{S})^{\times 2}$. Thus, the commutativity of the above diagram shows that

$$\ker(\pi^*)_{\mathrm{H}^1}\bigcap\ker\sum_t\operatorname{Cor}_{\mathbf{k}(t)/k}\simeq\ker\left(\mathrm{N}\colon\bigoplus_{t\in\mathcal{S}}\langle\varepsilon_t\rangle\to k^\times/k^{\times 2}\right).$$

In particular, the image of $\ker\left(\mathrm{N}\colon \bigoplus_{t\in\mathcal{S}}\langle\varepsilon_t\rangle \to k^\times/k^{\times 2}\right)$ under β is contained inside of $\mathrm{Br}(\mathcal{G})$. Further, since π^*_{Br} is surjective, the image of $\ker\left(\mathrm{N}\colon \bigoplus_{t\in\mathcal{S}}\langle\varepsilon_t\rangle \to k^\times/k^{\times 2}\right)$ under β generates $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$.

It remains to understand which subsets $\mathcal{T}\subset\mathcal{S}$ give rise to $\beta_{\mathcal{T}}\in\mathrm{Br}_0(\mathcal{G})$. If $\beta_{\mathcal{T}}\in\mathrm{Br}_0(\mathcal{G})$, then by definition of $\mathrm{Br}_0(\mathcal{G})$ there exists $\mathcal{A}\in\mathrm{Br}(k)$ such that $\gamma(\varepsilon_{\mathcal{T}})-\mathcal{A}\in\ker(\pi^*)_{\mathrm{Br}}$. By (5.3), the kernel of $(\pi^*)_{\mathrm{Br}}$ is generated by $[\mathcal{G}_\eta]$. Thus, $\gamma(\varepsilon_{\mathcal{T}})=[\mathcal{G}_\eta]+\mathcal{A}$ or $\gamma(\varepsilon_{\mathcal{T}})=\mathcal{A}$, where both equalities are in $\mathrm{Br}(\mathbb{P}^1-\mathcal{S})$. The final statement of the proposition follows from these equalities after computing residues and evaluating at ∞. $\square$

5.2. **Clifford algebras and Brauer classes.** For a quadratic form F over a field of characteristic not equal to 2 we use $\mathrm{Clif}(F)$ to denote the Clifford algebra of the restriction of F to a maximal regular subspace, and $\mathrm{Clif}_0(F)$ to denote the corresponding even subalgebra. By Witt's Theorem [Lam05, Chap. I, Theorems 4.2 and 4.3], these do not depend on the choice of maximal regular subspace. If F has even rank, then $\mathrm{Clif}(F)$ is a central simple algebra, which will be identified with its class in the Brauer group. This extends to quadratic forms over finite étale algebras in the obvious way.

In particular, we will consider $\mathrm{Clif}(Q_{\mathcal{T}})\in\mathrm{Br}(\mathbf{k}(\mathcal{T}))$ where $Q_{\mathcal{T}}$ is a quadratic form defining the quadric $\mathcal{Q}_{\mathcal{T}}$ corresponding to a subscheme $\mathcal{T}\subset\mathcal{S}$. This depends on the choice of quadratic form as indicated by the following lemma.

Lemma 5.4. *Let $s\in\mathcal{S}$ and $c\in\mathbf{k}(s)^\times$. Then*

$$\mathrm{Clif}(cQ_s)=\mathrm{Clif}(Q_s)+(\varepsilon_s,c)\in\mathrm{Br}(\mathbf{k}(s))\,.$$

Proof. This follows from a short calculation using [Lam05, Chap. V, Corollary 2.7]. $\square$

For a rank 4 quadric $\mathcal{Q}_s, s\in\mathcal{S}$ with $\varepsilon_s\in\mathbf{k}(s)^{\times 2}$, any quadratic form Q_s defining $\mathcal{Q}_s$ is a constant multiple of the reduced norm form of a quaternion algebra whose class in $\mathrm{Br}(\mathbf{k}(s))$ is equal to $\mathrm{Clif}(Q_s)$ [EKM08, Prop. 12.4]. The following lemma gives a description of $\mathrm{Clif}(Q_s)$ in cases when $\varepsilon_s\notin\mathbf{k}(s)^{\times 2}$.

Lemma 5.5. *Assume that k is a field of characteristic different from 2 and that $s\in\mathcal{S}$ with $\varepsilon_s\notin\mathbf{k}(s)^{\times 2}$ such that $\mathcal{Q}_s$ has a smooth $\mathbf{k}(s)$-point. Let Q_s be a quadratic form whose vanishing definines $\mathcal{Q}_s$. Then for any $\mathrm{Gal}(\mathbf{k}(s))$-stable pair $\{x,x'\}\subset\mathcal{Q}_s(\overline{k})$ and any $\mathbf{k}(s)$-linear form ℓ defining a hyperplane tangent to $\mathcal{Q}_s$ at a smooth point with $\ell(x)\ell(x')\neq 0$ we have the following equality in $\mathrm{Br}(\mathbf{k}(s))$:*

$$\mathrm{Clif}(Q_s)=\left(\varepsilon_s,-\frac{B_{Q_s}(x,x')}{\ell(x)\ell(x')}\right)\,.$$

Proof. By [VAV14, Lemma 2.1], for any $\ell=\ell_0$ tangent to $\mathcal{Q}_s$ at a smooth point, the quadric $\mathcal{Q}_s$ is defined by the vanishing of $Q_s=c(\ell_0\ell_1-\ell_2^2+\varepsilon_s\ell_3^2)$, for some linear forms ℓ_1,ℓ_2,ℓ_3 and some $c\in\mathbf{k}(s)^\times$. In particular, we have $\ell_0(x)\ell_1(x)=\ell_2(x)^2-\varepsilon_s\ell_3(x)^2$ and similarly for x'.

Thus, we may compute:

$$
\begin{aligned}
-\frac{B_{Q_s}(x,x')}{\ell(x)\ell(x')} &= -c\cdot\frac{\ell_0(x)\ell_1(x')+\ell_0(x')\ell_1(x)-2\ell_2(x)\ell_2(x')+2\varepsilon_s\ell_3(x)\ell_3(x')}{\ell_0(x)\ell_0(x')}\\
&= -c\left(\frac{\ell_2(x')^2-\varepsilon_s\ell_3(x')^2}{\ell_0(x')^2}+\frac{\ell_2(x)^2-\varepsilon_s\ell_3(x)^2}{\ell_0(x)^2}-2\frac{\ell_2(x)\ell_2(x')}{\ell_0(x)\ell_0(x')}+2\varepsilon_s\frac{\ell_3(x)\ell_3(x')}{\ell_0(x)\ell_0(x')}\right)\\
&= -c\left[\left(\frac{\ell_2(x)}{\ell_0(x)}-\frac{\ell_2(x')}{\ell_0(x')}\right)^2-\varepsilon_s\left(\frac{\ell_3(x)}{\ell_0(x)}-\frac{\ell_3(x')}{\ell_0(x')}\right)^2\right],
\end{aligned}
$$

which shows that $\left(\varepsilon_s, -\frac{B_{Q_s}(x,x')}{\ell(x)\ell(x')}\right) = (\varepsilon_s, -c)$. Thus, it remains to relate the quaternion algebra $(\varepsilon_s, -c)$ to the Clifford algebra of Q_s. By [Lam05, Chap. V, Corollary 2.7],

$$\mathrm{Clif}(Q_s)\simeq \mathrm{Clif}(Q_s|_{\langle \ell_0,\ell_1\rangle})\otimes \mathrm{Clif}(c^2\cdot Q_s|_{\langle \ell_2,\ell_3\rangle})\simeq \mathrm{M}_2(k)\otimes(-c, c\varepsilon_s).$$

To complete the proof, we observe that $(-c, c\varepsilon_s) = (-c,\varepsilon_s) = (\varepsilon_s, -c)\in \mathrm{Br}(k)$. ☐

Definition 5.6. *Given $\mathcal{T}\subset\mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$, define*

$$\mathcal{C}_{\mathcal{T}} := \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}(\mathrm{Clif}(Q_{\mathcal{T}}))\in \mathrm{Br}(k).$$

Remark 5.7. Even though $\mathrm{Clif}(Q_{\mathcal{T}})$ may depend on the choice of quadratic form defining the pencil, the condition $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$ ensures that the class $\mathcal{C}_{\mathcal{T}}$ does not. Indeed, if one computes $\mathcal{C}_{\mathcal{T}}$ using instead a form $cQ_{\mathcal{T}}$ which differs from $Q_{\mathcal{T}}$ by $c\in k^\times$, Lemma 5.4 shows that the result will differ by $\mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}, c) = (\mathrm{N}(\varepsilon_{\mathcal{T}}), c)$, which is trivial whenever $\mathrm{N}(\varepsilon_{\mathcal{T}})$ is a square.

Lemma 5.8. *The kernel of the canonical map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is generated by*

$$\{\mathcal{C}_s \ :\ s\in\mathcal{S} \text{ such that } \varepsilon_s\in\mathbf{k}(s)^{\times 2}\}.$$

Proof. By the exact sequence of low degree terms coming from the Hochschild-Serre spectral sequence, the kernel of $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is the image of the cokernel $\mathrm{Pic}(X)\to\mathrm{Pic}(\overline{X})^{G_k}$. By [VAV14, Prop. 2.3] (which relies on results from [KST89]), $\mathrm{Pic}(\overline{X})^{G_k}$ is freely generated by the hyperplane section and, for every $s\in\mathcal{S}$ such that $\varepsilon_s\in\mathbf{k}(s)^{\times 2}$, the divisor class $\mathrm{Norm}_{\mathbf{k}(s)/k}([C_s])$ where C_s is obtained by intersecting X with a plane contained in $\mathcal{Q}_s$. Since the hyperplane section is k-rational, the cokernel of $\mathrm{Pic}(X)\to\mathrm{Pic}(\overline{X})^{G_k}$ is generated by

$$\left\{\mathrm{Norm}_{\mathbf{k}(s)/k}([C_s]) \ :\ s\in\mathcal{S} \text{ such that } \varepsilon_s\in\mathbf{k}(s)^{\times 2} \text{ and } \mathcal{Q}_s \text{ contains no } k\text{-rational planes}\right\}.$$

By definition, the image of $[C_s]$ in $\mathrm{Br}(\mathbf{k}(s))$ is the Severi-Brauer variety whose points parametrize representatives of the class $[C_s]$. Since ε_s is a square, by [CTS93, Thm. 2.5], $\mathcal{Q}_s\cap H\cong Z_s\times Z_s$ for the conic Z_s that is a smooth hyperplane section of $\mathcal{Q}_s\cap H$. Since planes contained in $\mathcal{Q}_s$ in a fixed ruling are uniquely determined by points on a smooth plane intersected with $\mathcal{Q}_s$, we deduce that $[C_s]\mapsto Z_s\in\mathrm{Br}(\mathbf{k}(s))$. By [EKM08, Prop. 12.4] we also have that $\mathrm{Clif}(Q_s) = Z_s\in\mathrm{Br}(\mathbf{k}(s))$. Hence, $\mathrm{Norm}_{\mathbf{k}(s)/k}([C_s]) = \mathrm{Cor}_{\mathbf{k}(s)/k}(\mathrm{Clif}(Q_s)) = \mathcal{C}_s$. ☐

5.3. **Local evaluation maps.**

Lemma 5.9. *Assume that k is a field of characteristic different from 2. If there exists a degree 2 subscheme $\mathcal{T}\subset\mathcal{S}$ such that for all $t\in\mathcal{T}$, $\varepsilon_t\in\mathbf{k}(t)^{\times 2}$ and $\mathcal{Q}_t$ has a smooth k-point, then $X(k)\neq\emptyset$.*

Proof. Let $\mathcal{T}(\bar{k}) = \{t_1, t_2\}$. The assumptions in the lemma imply that there are $k(t_i)$-rational planes contained in $\mathcal{Q}_{t_i}$. The intersection of one with X gives a $k(t_i)$-rational conic C_i on X. After possibly choosing a different plane for t_2, we may assume the pair $\{C_1, C_2\}$ are Galois invariant. As computed in [VAV14, Proof of Proposition 2.2] we have $C_1.C_2 = 1$. Therefore the intersection of these divisors produces a k-point on X. □

Lemma 5.10. *Assume that k is a local field of chararcteristic different from 2 and let $\mathcal{T} \subset \mathcal{S}$ be a degree 2 subscheme such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. Then, for any quadratic extension K/k with $\varepsilon_{\mathcal{T}} \in \mathbf{k}(\mathcal{T}_K)^{\times 2}$ and $K \neq \mathbf{k}(\mathcal{T})$, there exists $y \in \mathcal{G}(k)$ corresponding to a quadratic point $\operatorname{Spec} K \to X$. Moreover, for such y,*

$$\beta_{\mathcal{T}}(y) = \begin{cases} \mathcal{C}_{\mathcal{T}} & \text{if } \varepsilon_{\mathcal{T}} \notin \mathbf{k}(T)^{\times 2}, \\ 0 & \text{if } \varepsilon_{\mathcal{T}} \in \mathbf{k}(T)^{\times 2}. \end{cases}$$

Proof. If $X(k) \neq \emptyset$, then for any nontrivial extension K/k we have $X(k) \subsetneq X(K)$ because k is local (see, e.g., [LL18, Proposition 3.8]). Then any pair of $\mathrm{Gal}(K/k)$-conjugate points on X will give the required $y \in \mathcal{G}(k)$. Now we prove the first statement in the case where $X(k) = \emptyset$. Over any local field, there is a unique rank 4 quadric (up to isomorphism) that fails to have a point, and it has square discriminant. Thus, for any $t \in \mathcal{T}$, if $\varepsilon_t \notin \mathbf{k}(t)^{\times 2}$, then $\mathcal{Q}_t(\mathbf{k}(t))$ contains a smooth point. Hence, Lemma 5.9 gives the existence of K-points on X for any K such that $\varepsilon_{\mathcal{T}} \in \mathbf{k}(\mathcal{T}_K)^{\times 2}$.

If $\varepsilon_t \in \mathbf{k}(t)^{\times 2}$ for some (equivalently, for all) $t \in \mathcal{T}$, then $\mathcal{Q}_t$ may not have a smooth $\mathbf{k}(t)$-point, but it will have a smooth point over any quadratic extension of $\mathbf{k}(t)$. If K/k is a quadratic extension different from $\mathbf{k}(\mathcal{T})/k$, then $\mathbf{k}(\mathcal{T}_K)$ will be a quadratic extension of $\mathbf{k}(\mathcal{T})$ and hence we may again apply Lemma 5.9.

Now suppose y corresponds to the line joining the K/k-conjugate points $x, x' \in X(K)$, with K satisfying the conditions of the lemma. By continuity of the evaluation map, we may reduce to the case where $\pi(y) \neq \infty$. Then $\pi(y) = -B_0(x,x')/B_\infty(x,x')$, and so

$$\pi(y) - \theta = -\frac{B_0(x,x') + \theta B_\infty(x,x')}{B_\infty(x,x')} = -\frac{B_{\mathcal{S}}(x,x')}{B_\infty(x,x')}\,.$$

Using this we compute

$$\begin{aligned}
\beta_{\mathcal{T}}(y) &= \mathrm{Cor}_{\mathbf{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{T}}, \pi(y) - \theta) = \mathrm{Cor}_{\mathbf{k}(\mathcal{S})/k}\left(\varepsilon_{\mathcal{T}}, -B_{\mathcal{S}}(x,x')/B_\infty(x,x')\right) \\
&= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\left(\varepsilon_{\mathcal{T}}, -B_{\mathcal{T}}(x,x')\right) + \left(\mathrm{N}_{\mathbf{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}), B_\infty(x,x')\right) \\
&= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\left(\varepsilon_{\mathcal{T}}, -B_{\mathcal{T}}(x,x')\right) && (\text{since } N(\varepsilon_{\mathcal{T}}) \in k^{\times 2}\,) \\
&= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\left[(\varepsilon_{\mathcal{T}}, \ell_{\mathcal{T}}(x)\ell_{\mathcal{T}}(x')) + \mathrm{Clif}(Q_{\mathcal{T}})\right] && (\text{by Lemma } 5.5) \\
&= \mathrm{Cor}_{\mathbf{k}(\mathcal{T}_K)/k}(\varepsilon_{\mathcal{T}}, \ell_{\mathcal{T}}(x)) + \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\mathrm{Clif}(Q_{\mathcal{T}}) \\
&= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\mathrm{Clif}(Q_{\mathcal{T}}) && (\text{since } \varepsilon_{\mathcal{T}} \in \mathbf{k}(\mathcal{T}_K)^{\times 2}\,) \\
&= \mathcal{C}_{\mathcal{T}} && (\text{by Definition } 5.6)\,.\square
\end{aligned}$$

Lemma 5.11. *Assume that k is a local field of characteristic not equal to 2. Suppose $s \in \mathcal{S}(k)$ is such that $\mathcal{Q}_s$ has a smooth k-point and let v_s denote the vertex of $\mathcal{Q}_s$. For any $t \in \mathbb{A}^1(k) - \{s\}$ sufficiently close to s, we have*

$$\mathcal{G}_t(k) \neq \emptyset \iff (\varepsilon_s, t - s) = \mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_\infty(v_s)) \text{ in } \mathrm{Br}(k)\,.$$

Remark 5.12. Note that by Lemma 5.4, the sum $\mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_\infty(v_s))$ appearing on the right-hand side above does not depend on the choice of quadratic form defining the pencil.

Proof. Since $t \in \mathbb{A}^1(k) - \{s\}$ is sufficiently close to s and $\mathcal{S}$ is closed, we have $t \notin \mathcal{S}$ and Q_t has rank 5. So by [EKM08, Ex. 85.4] the Severi-Brauer variety $\mathcal{G}_t$ and the even Clifford algebra $\mathrm{Clif}_0(Q_t)$ (which is a central simple k-algebra) determine the same class in $\mathrm{Br}(k)$. In particular, $\mathcal{G}_t(k) \neq \emptyset$ if and only if $\mathrm{Clif}_0(Q_t) = 0$ in $\mathrm{Br}(k)$.

The quadratic forms Q_t and $Q_t|_{\langle v_s\rangle^\perp} \perp Q_t|_{\langle v_s\rangle}$ are equivalent by [Lam05, Chap. I, Cor 2.5]. Therefore,

$$\begin{aligned}\mathrm{Clif}_0(Q_t) &= \mathrm{Clif}(-Q_t(v_s)\cdot Q_t|_{\langle v_s\rangle^\perp}) && \text{(by [Lam05, Chap. V, Cor. 2.9])}\\ &= \mathrm{Clif}(Q_t|_{\langle v_s\rangle^\perp}) + (\mathrm{disc}(Q_t|_{\langle v_s\rangle^\perp}), -Q_t(v_s)) && \text{(by Lemma 5.4).}\end{aligned}$$

For t sufficiently close to s, the quadratic forms $Q_t|_{\langle v_s\rangle^\perp}$ and $Q_s|_{\langle v_s\rangle^\perp}$ will be equivalent. For such t, $\mathrm{Clif}(Q_t|_{\langle v_s\rangle^\perp}) = \mathrm{Clif}(Q_s) \in \mathrm{Br}(k)$ and $\mathrm{disc}(Q_t|_{\langle v_s\rangle^\perp}) = \mathrm{disc}(Q_s) \in k^\times/k^{\times 2}$. Hence

$$\mathrm{Clif}_0(Q_t) = \mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_t(v_s))\,.$$

To complete the proof, we note that $Q_t(v_s) = Q_\infty(v_s)(t-s)$. $\square$

Lemma 5.13. *Assume that k is a local field and $\mathcal{T} \subset \mathcal{S}$ is a degree 2 subscheme with $\mathrm{N}(\varepsilon_\mathcal{T}) \in k^{\times 2}$ and $\varepsilon_\mathcal{T} \notin \mathbf{k}(\mathcal{T})^{\times 2}$. Then there exists $y \in \mathcal{G}_\mathcal{T}(\mathbf{k}(\mathcal{T}))$ such that $\pi(y) = \mathcal{T}$. Moreover, for any such y,*

$$\mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}(\beta_\mathcal{T}(y)) = \mathcal{C}_\mathcal{T} + (\varepsilon, -\Delta_\mathcal{T}\,\mathrm{N}(Q_\infty(v_\mathcal{T}))) \in \mathrm{Br}(k)\,,$$

where $\varepsilon \in k^\times$ is an element whose image in $\mathbf{k}(\mathcal{T})^\times/\mathbf{k}(\mathcal{T})^{\times 2}$ represents $\varepsilon_\mathcal{T}$, $\Delta_\mathcal{T}$ is the discriminant of $\mathbf{k}(\mathcal{T})/k$ (which we take to be 1 if $\mathcal{T}$ is reducible), and $v_\mathcal{T}$ is the vertex of $\mathcal{Q}_\mathcal{T}$.

Proof. By Corollary 5.2(5) there exists $\varepsilon \in k^\times$ such that $\varepsilon \cdot \varepsilon_t \in \mathbf{k}(t)^{\times 2}$ for all $t \in \mathcal{T}$. Fix a closed point $s \in \mathcal{T}$, and let s' be the unique $\mathbf{k}(s)$ point in $\mathcal{T}_{\mathbf{k}(s)} - \{s\}$.

Since $\varepsilon_s \notin \mathbf{k}(s)^{\times 2}$, $\mathcal{Q}_s$ is a cone over an isotropic quadric and as such contains smooth $\mathbf{k}(s)$-points and many $\mathbf{k}(s)$-rational lines (passing through the vertex). Hence $\mathcal{G}_s(\mathbf{k}(s))$ is nonempty. By the implicit function theorem, we can find $t \in (\mathbb{P}^1 - \{s\})(\mathbf{k}(s))$ arbitrarily close to s such that $\mathcal{G}_t(\mathbf{k}(s)) \neq \emptyset$. In addition, by Lemma 5.11 and the fact that $\beta\colon \mathcal{G}(\mathbf{k}(s)) \to \mathrm{Br}(\mathbf{k}(s))$ is locally constant, we may choose such a t sufficiently close to s so that

(1) $(\varepsilon, t-s) = \mathrm{Clif}(Q_s) + (\varepsilon, -Q_\infty(v_s)) \in \mathrm{Br}(\mathbf{k}(s))$,
(2) $\beta_\mathcal{T}(\mathcal{G}_s(\mathbf{k}(s))) = \beta_\mathcal{T}(\mathcal{G}_t(\mathbf{k}(s))) \in \mathrm{Br}(\mathbf{k}(s))$, and
(3) $t - s'$ and $s - s'$ represent the same class in $\mathbf{k}(s)^{\times 2}$.

Then for $y_1 \in \mathcal{G}_s(\mathbf{k}(s))$ and $y_1' \in \mathcal{G}_t(\mathbf{k}(s))$, we have

$$\beta_\mathcal{T}(y_1) = \beta_\mathcal{T}(y_1') = (\varepsilon, (t-s)(t-s')) = \mathrm{Clif}(Q_s) + (\varepsilon, -Q_\infty(v_s)(s-s'))\,.$$

If $y\colon \mathrm{Spec}(\mathbf{k}(\mathcal{T})) \to \mathcal{G}$ is such that $\pi(y) = \mathcal{T}$ it follows that

$$\begin{aligned}\mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}(\beta_\mathcal{T}(y)) &= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\left[\mathrm{Clif}(Q_\mathcal{T}) + (\varepsilon, -Q_\infty(v_\mathcal{T})) + (\varepsilon, s-s')\right]\\ &= \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}\left(\mathrm{Clif}(Q_\mathcal{T})\right) + (\varepsilon, \mathrm{N}(Q_\infty(v_\mathcal{T}))) + (\varepsilon, (s-s')(s'-s))\\ &= \mathcal{C}_\mathcal{T} + (\varepsilon, \mathrm{N}(Q_\infty(v_\mathcal{T}))) + (\varepsilon, -\Delta_\mathcal{T})\ .\end{aligned}$$

$\square$

5.4. **Evaluation of Brauer classes on $\mathcal{G}(\mathbb{A}_k)$.**

Definition 5.14. *Let k be a global field of characteristic not equal to 2. Given $\mathcal{T} \subset \mathcal{S}$ define*

$$R_{\mathcal{T}} := \{v \in \Omega_k : \varepsilon_{\mathcal{T}_v} \in \mathbf{k}(\mathcal{S}_v)^{\times 2} \text{ and } \mathcal{C}_{\mathcal{T}_v} \neq 0\}.$$

Theorem 5.15. *Assume that k is a global field of characteristic different from 2.*

(1) There exists $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ such that for all degree 2 subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, we have $\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \frac{\#R_{\mathcal{T}}}{2} \in \mathbb{Q}/\mathbb{Z}$.

(2) For all $t \in \mathcal{S}(k)$ there exists $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ such that for all degree 2 subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $t \in \mathcal{T}$, we have $\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \frac{\#R_t}{2} \in \mathbb{Q}/\mathbb{Z}$.

Proof. (1) It suffices to prove the result for $\{\beta_{\mathcal{T}} : \mathcal{T} \in \mathbb{T}\}$, where $\mathbb{T}$ is a collection of degree 2 subschemes of $\mathcal{S}$ as in Corollary 5.2(5), with corresponding $\varepsilon \in k^\times$ simultaneously representing the discriminants of all $\mathcal{T} \in \mathbb{T}$.

We define an adelic point $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ as follows. For $v \in \Omega_k$ such that $\varepsilon \in \mathbf{k}(\mathcal{T}_v)^{\times 2}$ for some (equivalently all) $\mathcal{T} \in \mathbb{T}$, let $y_v \in \mathcal{G}(k_v)$ be any point (which exists by Corollary 4.2). Note that if $\varepsilon \in \mathbf{k}(\mathcal{T}_v)^{\times 2}$, then $\beta_{\mathcal{T}} \otimes k_v = 0$ by Proposition 5.1. For each $v \in \Omega_k$ with $\varepsilon \notin \mathbf{k}(\mathcal{T}_v)^{\times 2}$ for some (equivalently all) $\mathcal{T} \in \mathbb{T}$, let $y_v \in \mathcal{G}(k_v)$ be a point corresponding to a $k_v(\sqrt{\varepsilon})$-point on X, as provided by Lemma 5.10. Note that Lemma 5.10 further implies that for such y_v, $\beta_{\mathcal{T}}(y_v) = \mathcal{C}_{\mathcal{T}_v}$ for all $\mathcal{T} \in \mathbb{T}$. Thus, for any $\mathcal{T} \in \mathbb{T}$ we have

$$\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon\notin\mathbf{k}(\mathcal{T}_v)^{\times 2}} \mathrm{inv}_v\left(\mathcal{C}_{\mathcal{T}_v}\right) = \sum_{\varepsilon\in\mathbf{k}(\mathcal{T}_v)^{\times 2}} \mathrm{inv}_v\left(\mathcal{C}_{\mathcal{T}_v}\right) = \frac{\#R_{\mathcal{T}}}{2} \in \mathbb{Q}/\mathbb{Z}\,,$$

where the penultimate equality follows from quadratic reciprocity.

(2) Let $t \in \mathcal{S}(k)$ and set $\varepsilon := \varepsilon_t$. If t is not contained in any degree 2 subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, then we need only show that $\mathcal{G}(\mathbb{A}_k) \neq \emptyset$, which follows from Corollary 4.2. Thus, we may assume there is some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ containing t such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For any such $\mathcal{T}$ we have $\varepsilon_{\mathcal{T}} = (\varepsilon, \varepsilon) \in \mathbf{k}(\mathcal{T})^\times/\mathbf{k}(\mathcal{T})^{\times 2} \simeq k^\times/k^{\times 2} \times k^\times/k^{\times 2}$.

We define an adelic point $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ as follows. For $v \in \Omega_k$ such that $\varepsilon \in k_v^{\times 2}$, take y_v to be any point of $\mathcal{G}(k_v)$ (which exists by Corollary 4.2). For $v \in \Omega_k$ such that $\varepsilon \notin k_v^{\times 2}$ we take $y_v \in \mathcal{G}(k_v)$ to be any point such that $\pi(y_v) \in \mathbb{P}^1(k_v)$ is close enough t so that Lemma 5.11 applies (note that $\mathcal{Q}_t$ is a cone over an isotropic quadric surface so the hypothesis of the Lemma 5.11 is satisfied) and so that, for all $s \in \mathcal{S}(k) - \{t\}$ with $\varepsilon\varepsilon_s \in k^{\times 2}$, $(\pi(y_v) - s)$ and $(t - s)$ have the same class in $k_v^\times/k_v^{\times 2}$.

Suppose $\mathcal{T} = \{s, t\} \subset \mathcal{S}(k)$ is such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For $v \in \Omega_k$ such that $\varepsilon \in k_v^{\times 2}$, we have $\mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = 0$. For $v \in \Omega_k$ such that $\varepsilon \notin k_v^{\times 2}$ we have

$$\begin{aligned}\mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) &= \mathrm{inv}_v\left(\varepsilon, (\pi(y_v) - t)(\pi(y_v) - s)\right)\\ &= \mathrm{inv}_v\left(\varepsilon, \pi(y_v) - t)\right) + \mathrm{inv}_v\left(\varepsilon, t - s\right)\\ &= \mathrm{inv}_v(\mathrm{Clif}(Q_t)) + \mathrm{inv}_v(\varepsilon, -Q_\infty(v_t)) + \mathrm{inv}_v(\varepsilon, t - s) \qquad \text{(By Lemma 5.11)}\,.\end{aligned}$$

Since $(\varepsilon, -Q_\infty(v_t)(t - s))$ is an element of $\mathrm{Br}(k)$, its local invariants sum to 0. Furthermore, $(\varepsilon, -Q_\infty(v_t)(t - s))$ has trivial invariant at all $v \in \Omega_k$ where $\varepsilon \in k_v^{\times 2}$. Thus,

$$\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon\notin k_v^{\times 2}} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon\notin k_v^{\times 2}} \mathrm{inv}_v(\mathrm{Clif}(Q_t)) = \sum_{\varepsilon\in k_v^{\times 2}} \mathrm{inv}_v(\mathrm{Clif}(Q_t)).$$

where the last equality follows from the fact that the local invariants of $\mathrm{Clif}(Q_t) \in \mathrm{Br}(k)$ sum to 0. For $v \in \Omega_k$ such that $\varepsilon_t \in k_v^{\times 2}$ we have $\mathrm{inv}_v(\mathrm{Clif}(Q_t)) = \mathrm{inv}_v(\mathcal{C}_{t_v})$. Hence,

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon \in k_v^{\times 2}} \mathrm{inv}_v(\mathrm{Clif}(Q_t)) = \frac{\# R_t}{2}\,, \in \mathbb{Q}/\mathbb{Z}\,. \qquad \square$$

The following lemma relates the set $R_{\mathcal{T}}$ to the condition given in (2b) of Theorem 1.2.

Lemma 5.16. *Let k be a global field of characteristic not equal to 2 and $\mathcal{T} \subset \mathcal{S}$ a degree 2 subscheme such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. Then $v \in R_{\mathcal{T}}$ if and only if there are an odd number of components of $\mathcal{Q}_{\mathcal{T}_v} = \cup_{t_v \in \mathcal{T}_v} \mathcal{Q}_{t_v}$ which have no smooth $\mathbf{k}(t_v)$-point.*

Proof. Let $v \in \Omega_k$. First suppose that $\varepsilon_{\mathcal{T}_v} \notin \mathbf{k}(\mathcal{S}_v)^{\times 2}$. Then $v \notin R_{\mathcal{T}}$ by definition. Note also that for all $t_v \in \mathcal{T}_v$, $\varepsilon_{t_v} \notin \mathbf{k}(t_v)^{\times 2}$ (a priori this must hold for some $t_v \in \mathcal{T}_v$; the stronger conclusion holds because $\mathcal{T}$ has degree 2 and $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$). Recall that there is a unique anisotropic quadratic form of rank 4 over any local field and that it has square discriminant. Hence, when $\varepsilon_{\mathcal{T}_v} \notin \mathbf{k}(t_v)^{\times 2}$, all components $\mathcal{Q}_{t_v}$ have smooth $\mathbf{k}(t_v)$-points.

Now suppose that $\varepsilon_{\mathcal{T}_v} \in \mathbf{k}(\mathcal{S}_v)^{\times 2}$. As above $\varepsilon_{t_v} \in \mathbf{k}(t_v)^{\times 2}$, for each $t_v \in \mathcal{T}_v$. Then the rank 4 quadratic forms Q_{t_v} are equivalent to constant multiples of the norm forms of the quaternion algebras $\mathrm{Clif}(Q_{t_v})$ (see [EKM08, Prop. 12.4]). In particular, $\mathcal{Q}_{t_v}$ has a smooth $\mathbf{k}(t_v)$-point if and only if $\mathrm{Clif}(Q_{t_v}) = 0 \in \mathrm{Br}(\mathbf{k}(t_v))$. The corestriction maps $\mathrm{Cor}_{\mathbf{k}(t_v)/k_v} : \mathrm{Br}(\mathbf{k}(t_v)) \to \mathrm{Br}(k_v)$ are isomorphisms, so $\mathcal{C}_{\mathcal{T}_v} = \sum_{t_v \in \mathcal{T}_v} \mathrm{Cor}_{\mathbf{k}(t_v)/k_v} \mathrm{Clif}(Q_{t_v})$ is nonzero if and only if there are an odd number of components of $\mathcal{Q}_{\mathcal{T}_v}$ with no smooth $\mathbf{k}(t_v)$-points. By definition $v \in R_{\mathcal{T}}$ if and only if $\mathcal{C}_{\mathcal{T}_v} \neq 0$. $\square$

Lemma 5.17. *Assume that k is a global field of characteristic different from 2 and suppose $\mathcal{T} \subset \mathcal{S}$ is irreducible of degree 2 such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For any $t \in \mathcal{T}(\mathbf{k}(\mathcal{T}))$, the cardinalities of the sets*

$$R_{\mathcal{T}} \subset \Omega_k \quad \text{and} \quad R_t \subset \Omega_{\mathbf{k}(\mathcal{T})}$$

have the same parity.

Proof. For a prime $v \in \Omega_k$, we have $\varepsilon_{\mathcal{T}} \in \mathbf{k}(\mathcal{T}_v)^{\times 2}$ if and only if $\varepsilon_t \in \mathbf{k}(t)_w^{\times 2}$ for all (equivalently some) $w \in \Omega_{\mathbf{k}(\mathcal{T})}$ with $w \mid v$. For such v we have

$$\mathrm{inv}_v(\mathcal{C}_{\mathcal{T}_v}) = \mathrm{inv}_v(\mathrm{Cor}_{\mathbf{k}(\mathcal{T})/k}(\mathrm{Clif}(Q_{\mathcal{T}}))) = \sum_{w \mid v} \mathrm{inv}_w(\mathrm{Clif}(Q_t)) = \sum_{w \mid v} \mathrm{inv}_w(\mathcal{C}_{t_w}).$$

In particular, $\mathcal{C}_{\mathcal{T}_v} \neq 0$ if and only if there are an odd number of primes $w \mid v$ with $\mathcal{C}_{t_w} \neq 0$. $\square$

6. Proofs of the Main Theorems

6.1. Corollaries of Theorem 5.15.

Corollary 6.1. *Assume that k is a global field of characteristic different from 2 and that either of the following conditions hold:*

1. *Every nontrivial element of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ can be represented by $\beta_{\mathcal{T}}$ for some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\# R_{\mathcal{T}}$ even; or*
2. *Every nontrivial element of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ can be represented by $\beta_{\mathcal{T}}$ for some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}(k)$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$.*

Then $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

Proof. If condition (1) holds, then the corollary follows from Theorem 5.15(1). Now assume condition (2) holds and (1) fails. Then there exists a nontrivial element of $\mathrm{Br}(\mathcal{G})$ of the form $\beta_{\{t,t'\}}$ with $t, t' \in \mathcal{S}(k)$ such that $R_{\{t,t'\}}$ has odd cardinality. Note that $R_{\{t,t'\}}$ is the symmetric difference of R_t and $R_{t'}$. Thus, interchanging t and t' if needed we may assume R_t has even cardinality. Theorem 5.15(2) then gives an adelic point orthogonal to all $\beta_{\mathcal{T}}$ such that $\mathcal{T}$ has degree 2, contains t and $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. The result follows since Corollary 5.2(4) shows that such $\beta_{\mathcal{T}}$ generate $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$. $\square$

Remark 6.2. If both conditions of Corollary 6.1 fail, then $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) \simeq \mathbb{Z}/2\mathbb{Z}$ and any $\beta_{\mathcal{T}}$ with $\mathcal{T}$ of degree 2 which represents the nontrivial class has $\mathcal{T}$ is irreducible. Thus, $\mathcal{S}$ must contain an irreducible degree 2 subscheme $\mathcal{T}$ such that

- $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$,
- $\varepsilon_{\mathcal{T}} \notin \mathbf{k}(\mathcal{T})^{\times 2}$,
- if $\#(\mathcal{S} - \mathcal{T})(k) = 3$, then $\varepsilon_t \notin k^{\times 2}$ for all $t \in \mathcal{S} - \mathcal{T}$, and
- $\# R_{\mathcal{T}}$ is odd, which in particular implies that $\mathcal{Q}_{\mathcal{T}}$ has no smooth $\mathbf{k}(\mathcal{T})$-points.

Corollary 6.3. *Assume that k is a global field of characteristic not equal to 2. Suppose there is a degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ such that $R_{\mathcal{T}}$ has odd cardinality. Then $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \mathcal{G}(\mathbb{A}_k)$ and there exists a quadratic extension K/k such that $X_K(\mathbb{A}_K) \neq X_K(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$. In particular, $\mathcal{G}$ does not satisfy weak approximation and X does not satisfy the Hasse principle over quadratic extensions of k.*

Proof. The first statement follows immediately from Theorem 5.15(1). For the second statement we construct K by approximating fixed quadratic extensions of k_v for the places $v \in S := \{v : X(k_v) = \emptyset \text{ or } \mathrm{inv}_v \circ \beta_{\mathcal{T}} \colon \mathcal{G}(k_v) \to \mathbb{Q}/\mathbb{Z} \text{ is nonzero}\}$. (In particular, by Lemma 5.10 and the definition of $\mathcal{C}_{\mathcal{T}}$, we will approximate K at every ramification place of $\mathcal{C}_{\mathcal{T}}$.) For such v, if $\varepsilon \notin k_v^{\times 2}$, then we fix $K_v := k_v(\sqrt{\varepsilon})$. If v is such that $\varepsilon \in k_v^{\times 2}$, then we let K_v be any quadratic extension such that $X(K_v) \neq \emptyset$. Then Lemma 5.10 implies that for every $v \in S$, there exists a $y_v \in \mathcal{G}(k_v)$ corresponding to a quadratic point $\operatorname{Spec} K_v \to X$ and for all such y_v, $\beta_{\mathcal{T}}(y_v) = \mathcal{C}_{\mathcal{T}_v}$ if $\varepsilon \notin k_v^{\times 2}$ and $\beta_{\mathcal{T}}(y_v) = 0$ otherwise. Furthermore, for $v \notin S$ (which necessarily means that $\mathcal{C}_{\mathcal{T}_v} = 0$), our assumptions imply that $X(K_v) \neq \emptyset$ and that $\beta_{\mathcal{T}}(y_v) = 0$ for all $y_v \in \mathcal{G}(k_v)$. Thus, for all $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ corresponding to an adelic quadratic point $\operatorname{Spec}(\mathbb{A}_K) \to X$ we have

$$\sum_v \mathrm{inv}_v\, \beta_{\mathcal{T}}(y_v) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v\, \beta_{\mathcal{T}}(y_v) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v\, \mathcal{C}_{\mathcal{T}_v} = \sum_{\varepsilon \in k_v^{\times 2}} \mathrm{inv}_v\, \mathcal{C}_{\mathcal{T}_v} = \frac{\# R_{\mathcal{T}}}{2} \in \mathbb{Q}/\mathbb{Z}.$$

By Proposition 2.6(2) and Corollary 5.2(7), this implies that $X_K(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$. $\square$

Example 6.4. *Let $\mathcal{G} \to \mathbb{P}^1$ be the fibration of Severi-Brauer threefolds corresponding to the pencil containing the quadrics given by the vanishing of the rank 4 forms*

$$Q_0 = x_0x_1 - x_2^2 + \varepsilon x_3^2, \text{ and}$$
$$Q_1 = ax_0^2 + bx_1^2 - abx_2^2 - \varepsilon x_4^2$$

where $a, b, \varepsilon \in k^\times$. Then $\mathcal{T} = \{0, 1\} \subset \mathcal{S}$ is a degree 2 subscheme with $\varepsilon_{\mathcal{T}} = (\varepsilon, \varepsilon)$. Hence, Corollaries 5.2 and 6.1 imply that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Note that $\mathcal{Q}_0$ has smooth k-points, so $R_{\mathcal{T}} = R_1 = \{v \in \Omega_k : \varepsilon \in k_v^{\times 2} \text{ and } \mathrm{inv}_v(a, b) \neq 0\}$. Clearly one can choose a, b, ε so that $R_{\mathcal{T}}$ has odd cardinality (e.g., for $k = \mathbb{Q}$, $a = 3, b = 7, \varepsilon = 2$ we have $R_{\mathcal{T}} = \{7\}$), in which case $\mathcal{G}$

has a Brauer-Manin obstruction to weak approximation and the base locus X of the pencil is a counterexample to the Hasse principle over some quadratic extension by Corollary 6.3.

If $4-ab \in k_v^{\times 2} - k^{\times 2}$ for some prime $v \in R_{\mathcal{T}}$ (which holds for the values indicated above), then there exists no quadratic extension K/k such that X_K is everywhere locally solvable and $\mathrm{Br}(X_K) = \mathrm{Br}_0(X_K)$. To see this first observe that $4-ab = \varepsilon_t$ is the discriminant of the rank 4 quadric $Q_t = \frac{1}{1-ab}(Q_1 - abQ_0)$ (here $t = 1/(1-ab) \in \mathcal{S}(k)$). Now note that if a prime $v \in R_{\mathcal{T}}$ splits in a quadratic extension K, then X_K is not locally soluble because $\mathcal{Q}_1$ has no smooth K_w-points for the primes $w \mid v$. On the other hand, Proposition 5.1 shows that $\beta_{\mathcal{T}} \otimes K \in \mathrm{Br}(X_K)$ lies in the subgroup $\mathrm{Br}_0(X_K)$ if and only if $\varepsilon \in K^{\times 2}$ (in which case $K = k(\sqrt{\varepsilon})$ and all primes of $R_{\mathcal{T}}$ split in K) or $\varepsilon_{\mathcal{S}-\mathcal{T}} \in \mathbf{k}(\mathcal{S}_K)^{\times 2}$ (in which case $K = k(\sqrt{4-ab})$ and so some prime of $R_{\mathcal{T}}$ splits in K by assumption). We conclude that if K/k is a quadratic extension such that $\beta_{\mathcal{T}} \otimes K \in \mathrm{Br}_0(X_K)$, then $X_K(\mathbb{A}_K) = \emptyset$.

Corollary 6.5. *Assume that k is a global field of characteristic different from 2. There is an adelic 0-cycle of degree 1 on $\mathcal{G}$ orthogonal to $\mathrm{Br}(\mathcal{G})$.*

Proof. We may assume that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ (for otherwise the Corollary holds immediately) and hence, that the hypothesis of Corollary 6.1 fails. As explained in Remark 6.2, this implies that there is an irreducible degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, $\varepsilon_{\mathcal{T}} \notin \mathbf{k}(\mathcal{T})^{\times 2}$ and $R_{\mathcal{T}}$ has odd cardinality. By Corollary 5.2(3), the existence of such an irreducible $\mathcal{T}$ implies that $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ has order 2. Moreover, if $t \in \mathcal{T}(\mathbf{k}(\mathcal{T}))$ then, by Lemma 5.17, the set $R_t \subset \Omega_{\mathbf{k}(\mathcal{T})}$ has odd cardinality. Thus, by Theorem 5.15 applied over $\mathbf{k}(\mathcal{T})$ we obtain an effective adelic 0-cycle of degree 2 over k, denote it by (z_v), such that $\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(z_v)) = 1/2$. If $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ is any adelic point (which exists by Corollary 4.2), then $(z_v - y_v)$ is an adelic 0-cycle of degree 1 and, since $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{G}(\mathbb{A}_k)^{\beta_{\mathcal{T}}} = \emptyset$, we have $\sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(z_v - y_v)) = \sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(z_v)) - \sum_{v\in\Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = 1/2 - 1/2 = 0$. ☐

Remark 6.6. In the cases not already covered by Corollary 6.1, the proof above hinges on constructing an adelic 0-cycle of degree 2 on $\mathcal{G}$ that is not orthogonal to the Brauer group. Lemma 5.13 can be used to give an alternative construction of such a 0-cycle. See Section 7.1.

6.2. **Proof of Theorem 1.1.** Let $X' \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over k. The intersection of X' with a suitable linear subspace will yield a smooth del Pezzo surface X of degree 4. If k is a local field, then the result follows from Theorem 4.1. It remains to consider the case that k is global. The variety $\mathcal{G}$ parameterizing lines on the quadrics in the pencil of quadrics containing X is birational to the symmetric square of X by Proposition 3.3. So it suffices to prove that $\mathcal{G}$ has index 1. By Corollary 6.5, $\mathcal{G}$ has an adelic 0-cycle of degree 1 orthogonal to the Brauer group. Since $\mathcal{G}$ is pencil of Severi-Brauer varieties, [CTSD94, Theorem 5.1] shows that, when k is a number field, $\mathcal{G}$ must have a 0-cycle of degree 1.

6.3. **Proof of Theorem 1.2.** Let $X' \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over k. As noted above, X' contains a smooth del Pezzo surface of degree 4 over k. So Theorem 1.2(1) follows from Theorem 4.1. We now assume we are in case (2) of the theorem. In particular, k is global.

We claim that X' contains a smooth del Pezzo surface X of degree 4 such that the corresponding Severi-Brauer pencil $\mathcal{G}$ has $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. In case (2a) we have $n \geq 5$. Then by [Wit07, Section 3.5] the intersection of X' with an appropriate linear subspace is a smooth del Pezzo surface X of degree 4 with $\mathrm{Br}(X) = \mathrm{Br}_0(X)$. Corollary 5.2(7) implies that the corresponding $\mathcal{G}$ has $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) = 0$, so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ by Corollary 6.1. Now we turn our attention to (2b). If L/k is a quadratic field extension and $\mathcal{Q} \subset \mathbb{P}^4$ is a rank 4 quadric such that $X = \cap_{\sigma\in\mathrm{Gal}(L/k)}\sigma(\mathcal{Q})$, then there exists a degree 2 point $\mathcal{T} \in \mathcal{S}$ with $\mathbf{k}(\mathcal{T}) = L$ and $\{\sigma(\mathcal{Q})\}_{\sigma\in\mathrm{Gal}(L/k)} = \{\mathcal{Q}_s\}_{s\in\mathcal{T}(\bar{k})}$. Thus, under the assumptions of (2b), for each irreducible degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, the geometric components of $\mathcal{Q}_{\mathcal{T}}$ (which are defined over L) each fail to have smooth local points at an even number of places of L. By Lemma 5.16 this implies that $R_{\mathcal{T}}$ has even cardinality. Thus, in either case, we may (as explained in Remark 6.2) apply Corollary 6.1 and deduce that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

When k is a number field, Serre proved, assuming Schinzel's hypothesis, that the Brauer-Manin obstruction is the only obstruction for fibrations of Severi-Brauer varieties (Serre's result is unpublished, but a more general result [CTSD94, Theorem 4.2] implies this result of Serre). Thus, assuming Schinzel's hypothesis we obtain a k-point on $\mathcal{G}$ and, consequently, a quadratic point on X. If we do not assume Schinzel's hypothesis, it is enough to find a quadratic extension K/k such that $X_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$. The existence of such a K follows from Proposition 2.6(4), since as noted in Corollary 5.2(7) the map $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ given by Proposition 2.6(1) is an isomorphism.

7. Complements and Remarks

7.1. **Remarks on the cases not covered by Theorem 1.2.** Suppose X is a smooth del Pezzo surface of degree 4 with corresponding Severi-Brauer pencil $\mathcal{G}$ over a global field k of characteristic not equal to 2 such that the conditions of Corollary 6.1 are not satisfied. As noted in Remark 6.2 this implies that $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is cyclic of order 2, with the nontrivial class represented by $\beta_{\mathcal{T}}$ for an irreducible subscheme $\mathcal{T} \subset \mathcal{S}$ of degree 2 with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ for which $\#R_{\mathcal{T}}$ is odd. By Corollary 6.3, $\beta_{\mathcal{T}}$ obstructs weak approximation and so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ if and only if there exists a prime $v \in \Omega_k$ such that the evaluation map $\beta_{\mathcal{T}} : \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is not constant.

Let $\mathcal{C}'_{\mathcal{T}} := \mathcal{C}_{\mathcal{T}} + (\varepsilon, -\Delta_{\mathcal{T}}\,\mathrm{N}(Q_\infty(v_{\mathcal{T}}))) \in \mathrm{Br}(k)$ be the class appearing in Lemma 5.13 and define

$$R'_{\mathcal{T}} := \{v \in \Omega_k \ : \ \varepsilon_{\mathcal{T}} \notin \mathbf{k}(\mathcal{S}_v)^{\times 2} \text{ and } \mathrm{inv}_v(\mathcal{C}'_{\mathcal{T}}) \neq 0\}\,.$$

Since $R_{\mathcal{T}}$ has odd cardinality, so too must $R'_{\mathcal{T}}$. In particular, $R'_{\mathcal{T}}$ is nonempty. If $\mathcal{T}_v$ is reducible for a prime $v \in R'_{\mathcal{T}}$, then Lemma 5.13 shows that the evaluation map $\beta_{\mathcal{T}} \colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is nonconstant and so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{G}(\mathbb{A}_k)^{\beta_{\mathcal{T}}} \neq \emptyset$. If $\mathcal{T}_v$ is irreducible at $v \in R'_{\mathcal{T}}$, then Lemma 5.13 shows that $\beta_{\mathcal{T}} \otimes \mathbf{k}(\mathcal{T}_v) \colon \mathcal{G}(\mathbf{k}(\mathcal{T}_v)) \to \mathrm{Br}(\mathbf{k}(\mathcal{T}_v))$ is nonconstant. Indeed the lemma gives a $\mathbf{k}(\mathcal{T}_v)$-point where $\beta_{\mathcal{T}} \otimes \mathbf{k}(\mathcal{T}_v)$ takes the nonzero value $\mathcal{C}'_{\mathcal{T}_v}$, but $\beta_{\mathcal{T}} \otimes \mathbf{k}(\mathcal{T}_v)$ takes the value 0 at any elements in the subset $\mathcal{G}(k_v) \subset \mathcal{G}(\mathbf{k}(\mathcal{T}_v))$. Unfortunately, this is not enough to conclude that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ because $\beta_{\mathcal{T}} \colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ can still be constant. Using the lemma below one can check that this occurs at $v = 5$ for the pencil of quadrics defined by

$$Q_0 = -55x_1^2 + 2x_1x_2 + x_3^2 + 5x_4^2 \quad \text{and} \quad Q_\infty = 33x_0^2 - 5x_1^2 - x_2^2 + 10x_3x_4\,.$$

We note, however, that in this example (and in all others with $R'_{\mathcal{T}} \neq \emptyset$ that we have considered) there is some prime $w \in \Omega_k$ (in this case $w = 2$) for which the evaluation map $\beta_{\mathcal{T}} \colon \mathcal{G}(k_w) \to \mathrm{Br}(k_w)$ is not constant and, hence, $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

Lemma 7.1. *If $v \in R'_{\mathcal{T}}$ is such that $\mathcal{T}_v$ is irreducible, k_v has odd residue characteristic and $X(\mathbf{k}(\mathcal{T}_v)) = \emptyset$, then $\beta_{\mathcal{T}} \colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is constant.*

Proof. Suppose $X(\mathbf{k}(\mathcal{T}_v)) = \emptyset$ and let $y \in \mathcal{G}(k_v)$. Then y corresponds to a quadratic point $\operatorname{Spec}(K) \to X$, with K/k_v a quadratic field extension such that $K \neq \mathbf{k}(\mathcal{T}_v)$. Since k_v has odd residue characteristic, $\mathbf{k}(\mathcal{T}_L)$ is the compositum of all quadratic extensions of k_v. In particular, it must contain a square root of $\varepsilon_{\mathcal{T}}$ (since $\varepsilon_{\mathcal{T}} \in k_v^\times \mathbf{k}(\mathcal{T}_v)^{\times 2}$). Therefore, Lemma 5.10 applies, and its conclusion shows that $\beta_{\mathcal{T}}(y)$ does not depend on y. □

In contrast, the following lemma shows that for X (in place of $\mathcal{G}$) nonconstancy of an evaluation map over an extension of k_v does imply nonconstancy over k_v.

Lemma 7.2. *Let X be a del Pezzo surface of degree 4 over a local field k such that $X(k) \neq \emptyset$. If $\alpha \in \operatorname{Br}(X)$ is such that $\operatorname{inv}_k \circ \alpha\colon X(k) \to \mathbb{Q}/\mathbb{Z}$ is constant, then for all finite extensions K/k, $\operatorname{inv}_K \circ \alpha_K \colon X(K) \to \mathbb{Q}/\mathbb{Z}$ is constant and equal to $[K:k](\operatorname{inv}_k \circ \alpha)$.*

Remark 7.3. In the case that k_v has odd residue characteristic, $\mathcal{T}_v$ is irreducible, $\varepsilon_{\mathcal{T}_v} \in k_v^{\times 2}$, and $\operatorname{inv}_v(\mathcal{C}'_{\mathcal{T}}) \neq 0$, Lemma 7.2 can be used to prove the converse of Lemma 7.1. Namely, if $\beta_{\mathcal{T}}$ is constant on k_v-points, then $X(\mathbf{k}(\mathcal{T}_v))$ must be empty.

Proof. Let $P \in X(k)$. By [SS91, Lemma 4.4] (which follows from [CTC79, Theorem C]), every 0-cycle of degree 0 on X is linearly equivalent to one of the form $Q - P$ for some $Q \in X(k)$. Therefore, for any closed point R on X, there is some $Q \in X(k)$ such that $R \sim Q + (\deg(R) - 1)P$. Since evaluation of Brauer classes factors through rational equivalence and by assumption $\alpha(P) = \alpha(Q)$, we see that $\operatorname{inv}_K \circ \alpha_K = [K:k](\operatorname{inv}_k \circ \alpha)$ for any extension K/k. □

Remarks 7.4.

(1) The result of [CTC79] used in the proof above shows that every 0-cycle of degree 1 on a conic bundle with 5 or fewer degenerate fibers is rationally equivalent to a rational point. The example mentioned just before Lemma 7.1 shows that this does not extend to more general Severi-Brauer bundles (at least over a p-adic fields). Indeed, evaluation of Brauer classes factors through rational equivalence and in the example there is a Brauer class on $\mathcal{G}$ which is nonconstant on 0-cycles of degree 1, but is constant on rational points.

(2) If X/k is a del Pezzo surface of degree 4 over a number field which is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction, then as shown in [CTP00, Section 3.5] there exists $\alpha \in \operatorname{Br}(X)$ such that $X(\mathbb{A}_k)^\alpha = \emptyset$ (a priori multiple elements of $\operatorname{Br}(X)$ might be required to give the obstruction). An immediate consequence of Lemma 7.2 is that over any odd degree extension K/k the same Brauer class will give an obstruction, i.e., $X_K(\mathbb{A}_K)^{\alpha_K} = \emptyset$. This answers a question posed in [CTP00, Remark 3, p. 95]. In particular, this shows that the conjecture that all failures of the Hasse principle for del Pezzo surfaces of degree 4 are explained by the Brauer-Manin obstruction is compatible with the theorems of Amer, Brumer and Springer [Ame76, Bru78, Spr56] which imply that an intersection of quadrics with index 1 has a rational point.

7.2. **A degree 4 del Pezzo surface with obstructions only over odd degree extensions.**

Proposition 7.5. *Let $X/\mathbb{Q}$ be the del Pezzo surface of degree 4 given by the vanishing of*

$$Q_0 = (x_0 + x_1)(x_0 + 2x_1) - x_2^2 + 5x_4^2, \quad \textit{and} \quad Q_1 = 2(x_0x_1 - x_2^2 + 5x_3^2).$$

For any finite extension $K/\mathbb{Q}$ we have $X_K(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$ if and only if $[K:\mathbb{Q}]$ is odd.

Proof. This surface was considered by Birch and Swinnerton-Dyer [BSD75] who showed that X is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction. It follows from Lemma 7.2 that for any K with $[K:\mathbb{Q}]$ odd, X_K is also a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

Since X is locally soluble over $\mathbb{Q}$, $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is generated by the image of $\mathrm{Br}(X)[2]$. The singular quadrics in the pencil lie above $\mathcal{S}(\overline{\mathbb{Q}}) = \{0, \pm 1, \frac{\pm 4\sqrt{2}+5}{7}\} \subset \mathbb{P}^1$ and the corresponding discriminants satisfy $\varepsilon_0 = \varepsilon_1 = 5$, $\varepsilon_{-1} = -1$ and $\mathrm{N}(\varepsilon_{(\pm 4\sqrt{2}+5)/7}) = -1$. For any $K/\mathbb{Q}$ linearly disjoint from $k_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{5})$, the restriction map induces an isomorphism $\mathrm{Br}(X)/\mathrm{Br}_0(X) \simeq \mathrm{Br}(X_K)/\mathrm{Br}_0(X_K)$ and so $X_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$ by Lemma 2.1(2). On the other hand, if $K/\mathbb{Q}$ is not linearly disjoint from k_1, we can check directly that $X(K) \neq \emptyset$. Indeed, K must contain $\mathbb{Q}(\sqrt{d})$ for some $d \in \{-1, \pm 2, \pm 5, \pm 10\}$. Over these quadratic fields one can exhibit points:

$$(1:1:1:0:\sqrt{-1})\,,\ (1:-2:2\sqrt{2}:\sqrt{2}:1)\,,\ (4:9:6:0:5\sqrt{-2})\,,\ (0:0:\sqrt{5}:1:1)\,,$$
$$(5:0:0:0:\sqrt{-5})\,,\ (2\sqrt{10}:-\sqrt{10}:0:2:0)\,,\ (0:\sqrt{-10}:0:0:2)\,.$$

□

7.3. A degree 4 del Pezzo surface with index 4.

Theorem 7.6. *There exists a del Pezzo surface X of degree 4 over a field k of characteristic 0 such that X has index 4.*

Proof. Let k_0 be an algebraically closed field of characteristic 0. For $i = 1, \dots, 2g$, set $k_i := k_{i-1}((t_i))$ and set $k := k_{2g}$. By a result of Lang and Tate [LT58, p. 678], if A/k_0 is an abelian variety of dimension g and n is an integer, then there exists a torsor under $A_k = A \times_{k_0} \mathrm{Spec}(k)$ of period n and index n^{2g}. In particular, if C/k_0 is any genus 2 curve, then there exists a torsor under the Jacobian $J = \mathrm{Jac}(C_k)$ of C_k of period 2 and index 16. Since C is defined over the algebraically closed field k_0, it has a rational Weierstrass point over k. As observed by Flynn [Fly09], and worked out in detail by Skorobogatov [Sko10], if J_λ is a 2-covering $\pi_\lambda : J_\lambda \to J$ (i.e., a twist of $[2] : J \to J$ corresponding to $\lambda \in \mathrm{H}^1(k, J[2])$), then there are morphisms

$$J_\lambda \leftarrow \tilde{J}_\lambda \to Z_\lambda \to X_\lambda\,,$$

where $\tilde{J}_\lambda \to J_\lambda$ is the blow up of J_λ at $\pi_\lambda^{-1}(0_J)$, Z_λ is the desingularized Kummer variety associated to J_λ and $Z_\lambda \to X_\lambda$ is a double cover of a del Pezzo surface of degree 4. In particular, there is a degree 4 morphism $\tilde{J}_\lambda \to X_\lambda$. So the index of X_λ is at least $\mathrm{index}(J_\lambda)/4$, which will equal 4 for suitable choice of λ by the aforementioned result of Lang and Tate. □

Theorem 7.7. *Suppose k is a number field and Y is a torsor of period 2 under the Jacobian of a hyperelliptic curve over k with a rational Weierstrass point. The index of Y divides 8.*

Proof. As in the proof of the previous theorem, the index of Y divides $4\,\mathrm{index}(X)$ for some del Pezzo surface X of degree 4. The result follows from Theorem 1.1. □

Remarks 7.8.

(1) The conclusion of Theorem 7.7 was known to hold by work of Clark [Cla, Theorems 2 and 3] when k is a p-adic field and when k is a number field and Y is locally soluble.

(2) Arguing as in the proof of the theorem we see that the Kummer variety Z_λ has index dividing 4 when k is local or global field. This is lower than one would expect, given that Z_λ is an intersection of 3 quadrics in $\mathbb{P}^5_k$.
(3) The result of Lang-Tate quoted in the proof above shows that over general fields of characteristic 0, there are examples where Z_λ and Y have index 8 and 16, respectively.
(4) In response to a preliminary report on this work by the authors, John Ottem suggested the following alternate proof of Theorem 7.6. Let Y be an intersection of two general $(2,2)$ divisors in $\mathbb{P}^3 \times \mathbb{P}^4$ over $\mathbb{C}$, so that, by the Lefschetz hyperplane theorem, restriction gives an isomorphism $\mathrm{H}^4(\mathbb{P}^3 \times \mathbb{P}^4, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(Y, \mathbb{Z})$. Note that the generic fiber of the first projection is a del Pezzo surface of degree 4 over $\mathbf{k}(\mathbb{P}^3)$. Hence any threefold $V \subset Y$ can be expressed as $aH_1^2 + bH_1H_2 + cH_2^2$, where H_i denotes the pullback of $\mathcal{O}(1)$ under the projection π_i. Then the degree of $V \to \mathbb{P}^3$ is given by

$$V.H_1^3 = V.H_1^3.X = (aH_1^2 + bH_1H_2 + cH_2^2).H_1^3.(2H_1 + 2H_2)^2,$$

which must be divisible by 4. Thus $Y_{\mathbf{k}(\mathbb{P}^3)}$ has index 4.

This construction suggested by Ottem generalizes to arbitrary complete intersections. Namely, given a sequence of degrees $(d_1, \ldots, d_r)$ and an ambient dimension N, one can consider an intersection of general $(d_1, d_1), (d_2, d_2), \ldots, (d_r, d_r)$ hypersurfaces in $\mathbb{P}^M \times \mathbb{P}^N$. If $M > N - r$, then the same argument as above yields a $(d_1, \ldots, d_r)$ smooth complete intersection $Y \subset \mathbb{P}^N_{\mathbf{k}(\mathbb{P}^M)}$ with index $d_1 d_2 \cdots d_r$.
(5) After viewing an early draft of this paper, Olivier Wittenberg shared a correspondence of his from 2013 that provides yet another construction that proves Theorem 7.6. Wittenberg uses degenerations to geometrically reducible quadrics to construct complete intersections of n quadrics with index 2^n [Wit13].

7.4. **The index of a degree d del Pezzo surface.** The following table gives sharp upper bounds for the indices of degree d del Pezzo surfaces over local fields, number fields and arbitrary fields of characteristic 0. The entries in the column $d = 4$ are a consequence of the results in this paper, while for $d \neq 4$, they can be deduced fairly easily from known results as described below.

d	9	8	7	6	5	4	3	2	1
k arbitrary	3	4	1	6	1	4 [Thm. 7.6]	3	2	1
k a number field	3	2	1	6	1	2 [Thm. 1.1]	3	2	1
k a local field	3	2	1	2 or 3	1	2 [Thm. 1.1]	3	2	1

When $d = 9$, Y is a Severi-Brauer surface and so the index of Y divides 3 and examples of index 3 exist whenever $\mathrm{Br}(k)$ contains an element of order 3.

When $d = 8$, $Y = \mathrm{Res}_{L/k}(C)$ is the restriction of scalars of a conic C/L defined over a degree 2 étale algebra L/k [Poo17, Prop. 9.4.12]. Since the conic has a point over some quadratic extension L'/L, the index of Y divides 4 and over general fields there are examples with index 4. Over local and global fields however, the index must divide 2. Indeed, in this case C will have a point over a quadratic extension L'/L of the form $L' = k' \otimes_k L$ for some quadratic extension k'/k. The universal property of restriction of scalars then gives $Y(k') \neq \emptyset$, showing that the index divides 2.

When $d = 7$, $Y(k) \neq \emptyset$ over any field k and so the index is always equal to 1. The same applies to $d = 1, 5$ (see, e.g., [Poo17, Thm 9.4.8 and Section 9.4.11]).

For $d = 6$, Y is determined by a $\mathrm{Gal}(L/k)$-stable triple of geometric points on a Severi-Brauer surface S/L over a quadratic étale algebra L/k such that if $S \not\simeq \mathbb{P}^2_L$ then the class of S in the Brauer group does not lie in the image of $\mathrm{Br}(k) \to \mathrm{Br}(L)$ [Cor05]. If k is a local field and L is a quadratic field extension, then the map $\mathrm{Br}(k) \to \mathrm{Br}(L)$ is an isomorphism, so either $S = \mathbb{P}^2_L$ (in which case Y has index dividing 2) or $L = k \times k$ in which case the index of Y divides 3. One can construct examples of index 6 over number fields, by arranging to have index 2 at one completion and index 3 at another.

For $d = 3$ and k local, index 1 implies the existence of a k-rational point [Cor76], and so a cubic surface without points over some local field has index 3. This gives examples of index 3 over number fields as well.

For $d = 2$, index 2 examples can be obtained by blowing up a degree 4 del Pezzo surface of index 2 at a quadratic point. By Theorems 1.1 and 1.2, any del Pezzo surface of degree 4 without points over a local field gives such an example. The surface considered in Section 7.2 gives an example over a number field.

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