ARITHMETIC OF DEL PEZZO SURFACES OF DEGREE 4 AND VERTICAL BRAUER GROUPS

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ABSTRACT. We show that Brauer classes of a locally solvable degree 4 del Pezzo surface X are vertical for some projection away from a plane $g\colon X \dashrightarrow \mathbb{P}^1$, i.e., that every Brauer class is obtained by pullback from an element of $\mathrm{Br}\,\mathbf{k}(\mathbb{P}^1)$. As a consequence, we prove that a Brauer class obstructs the existence of a k-rational point if and only if all k-fibers of g fail to be locally solvable, or in other words, if and only if X is covered by curves that each have no adelic points. Using work of Wittenberg, we deduce that for certain quartic del Pezzo surfaces with nontrivial Brauer group the algebraic Brauer-Manin obstruction is sufficient to explain all failures of the Hasse principle, conditional on Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups. The proof of the main theorem is constructive and gives a simple and practical algorithm, distinct from that in [BBFL07], for computing all classes in the Brauer group of X (modulo constant algebras).

1. INTRODUCTION

Let X be a smooth projective geometrically integral variety over a global field k; write $X(\mathbb{A}_k)$ for its adelic points, and let $\mathbf{k}(X)$ denote its function field. In 1970, Manin used the Brauer group $\mathrm{Br}(X) := \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ to define the Brauer set $X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)$. He proved that this set contains the k-rational points of X, and that it can thus obstruct the existence of rational points [Man71]. This obstruction is known as the Brauer-Manin obstruction.

In some examples, the Brauer group is "vertical", and it is possible to interpret the Brauer-Manin obstruction through rational maps. For instance, consider the degree 4 del Pezzo surface X over $\mathbb{Q}$ given as the complete intersection of the following two quadrics in $\mathbb{P}^4$:

$$
\begin{aligned}
x_3 x_4 &= x_2^2 - 5x_0^2, \\
(x_3 + x_4)(x_3 + 2x_4) &= x_2^2 - 5x_1^2.
\end{aligned}
\tag{1.1}
$$

Birch and Swinnerton-Dyer showed that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$ [BSD75, Thm. 3]. Furthermore, in this example, $\mathrm{Br}\,X \subseteq f^*(\mathrm{Br}\,\mathbf{k}(\mathbb{P}^1))$, where $f\colon X \dashrightarrow \mathbb{P}^1$ is the map $[x_0 : \cdots : x_4] \mapsto [x_3 : x_4]$. Together, these facts imply that $f^{-1}(t)(\mathbb{A}_{\mathbb{Q}}) = \emptyset$ for all $t \in \mathbb{P}^1(\mathbb{Q})$, and thus the map f witnesses that each adelic point of X is not arranged in a way that is globally compatible.

It is natural to ask for which classes of varieties X one is guaranteed to have a vertical Brauer group, i.e., that $\mathrm{Br}\,X \subseteq f^*(\mathrm{Br}\,\mathbf{k}(Y))$, for some dominant map $f\colon X \dashrightarrow Y$, and thus possibly have a geometric interpretation of the Brauer-Manin obstruction as above. We show that locally solvable del Pezzo surfaces of degree 4 have this property.

Theorem 1.1. *Let k be a global field of characteristic not 2, and let X be a del Pezzo surface of degree 4 over k such that $X(\mathbb{A}_k) \neq \emptyset$. Then for all $\mathcal{A} \in \mathrm{Br}\,X$, there exists a*

map $g = g_{\mathcal{A}}\colon X \dashrightarrow \mathbb{P}^1$, obtained by projecting from a plane, with at most two geometrically reducible fibers such that $\mathcal{A} \in g^*(\operatorname{Br} \mathbf{k}(\mathbb{P}^1))$. Moreover, there exists a map $f\colon X \dashrightarrow \mathbb{P}^n$, obtained by projecting from a linear space, such that $\operatorname{Br} X$ is vertical with respect to f, i.e.,

$$\operatorname{Br} X \subseteq f^*(\operatorname{Br} \mathbf{k}(\mathbb{P}^n)),$$

where $n = \dim_{\mathbb{F}_2}(\operatorname{Br} X / \operatorname{im} \operatorname{Br} k)$ [2].

Remarks 1.2.

(1) For a del Pezzo surface of degree 4, the group $\operatorname{Br} X / \operatorname{im} \operatorname{Br} k$ is isomorphic to either 0, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [Man74, SD93].

(2) Theorem 1.1 holds over all fields of characteristic not 2, after replacing the hypothesis that $X(\mathbb{A}_k) \neq \emptyset$ with an assumption on the rank 4 quadrics in a pencil associated to X; see §3 for the precise condition.

Detecting Brauer-Manin obstructions through the fibers of a map. As in the example above, the Brauer-Manin obstruction to the existence of k-points on X manifests itself through the fibers of a map $g\colon X \to \mathbb{P}^1$ furnished by Theorem 1.1.

Corollary 1.3. *Retain the notation from Theorem 1.1. Assume that k is a number field. Then $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$ if and only if there exists a map $g\colon X \dashrightarrow \mathbb{P}^1$, obtained by projecting from a plane, with at most two geometrically reducible fibers such that*

$$g^{-1}(t)(\mathbb{A}_k) = \emptyset \quad \forall t \in \mathbb{P}^1(k).$$

Furthermore, if $\operatorname{Br} X / \operatorname{im} \operatorname{Br} k = \langle \mathcal{A} \rangle \cong \mathbb{Z}/2\mathbb{Z}$, then

$$\overline{\{t \in \mathbb{P}^1(k) : g_{\mathcal{A}}^{-1}(t)(\mathbb{A}_k) \neq \emptyset \text{ and } g_{\mathcal{A}}^{-1}(t) \text{ smooth}\}} = g_{\mathcal{A}}(X(\mathbb{A}_k)^{\operatorname{Br}}),$$

where the closure takes place in the adelic topology.

Remark 1.4. A stronger statement is true. If $n > 0$, then the map $g\colon X \dashrightarrow \mathbb{P}^1$ is given by projecting away from a plane W. The proof of Corollary 1.3 shows that $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$ if and only if $(X \cap L)(\mathbb{A}_k) = \emptyset$ for any hyperplane L that contains W. Since $g^{-1}(t) \subseteq X \cap L$ for some such L, the corollary follows from this stronger statement.

The proof of the corollary follows from Theorem 1.1 together with work of Skorobogatov [Sko96], Colliot-Thélène, Skorobogatov, and Swinnerton-Dyer [CTSSD98b] (which builds on [CTS82, CTSD94]), and Colliot-Thélène and Poonen [CTP00].

Sufficiency of Brauer-Manin obstructions. A special case of an important conjecture in the study of rational points on varieties, due to Colliot-Thélène and Sansuc [CTS80], says that for a degree 4 del Pezzo surface X, we have $X(k) \neq \emptyset$ as soon as $X(\mathbb{A}_k)^{\operatorname{Br}} \neq \emptyset$. Assuming Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups for elliptic curves, Wittenberg proved the conjecture for a general such X over a number field k, as well as in many other cases [Wit07, Thm. 3.36]. All the cases handled by Wittenberg require that $\operatorname{Br} X = \operatorname{Br} k$, and the proof proceeds by showing that certain fibrations of genus 1 curves satisfy the Hasse principle.

More generally, building on ideas of Swinnerton-Dyer [SD95, BSD01] that were formalized in [CTSSD98, CT01], Wittenberg showed that, for certain genus 1 fibrations, a non-empty Brauer set implies the existence of a rational point, again conditional on Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups [Wit07, Théorème 1.1]. If $\operatorname{Br} X / \operatorname{im} \operatorname{Br} k \cong$

$\mathbb{Z}/2\mathbb{Z}$, then the generic fiber of the map f given in the proof of Theorem 1.1 is a curve of genus 1. Under suitable hypotheses on X we are thus able to apply [Wit07, Théorème 1.1] and derive the following result.

Theorem 1.5. *Let k be a number field and let X be a degree 4 del Pezzo surface of BSD-type (see §5.4 for the precise definition), e.g., a surface given as the complete intersection of the following two quadrics in $\mathbb{P}^4$*

$$cx_3x_4 = x_2^2 - \varepsilon x_0^2,$$
$$(x_3 + x_4)(ax_3 + bx_4) = x_2^2 - \varepsilon x_1^2,$$

for any $a, b, c \in k^\times$, $\varepsilon \in k^\times \setminus k^{\times 2}$, with $(a - b)(a^2 + b^2 + c^2 - ab - ac - bc) \neq 0$ and $ab, \varepsilon(a^2 + b^2 + c^2 - ab - ac - bc) \notin k^{\times 2}$. Assume Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups of elliptic curves. Then $X(k)$ is dense (in the adelic topology) in $X(\mathbb{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction to the Hasse principle and weak approximation on X is the only one.

Interestingly, to apply Wittenberg's theorem, we must fix f such that there are *exactly* 6 geometrically reducible fibers. This is in contrast to the condition that $g_{\mathcal{A}}$ has at most 2 geometrically reducible fibers, which is needed to prove Corollary 1.3. Indeed, this requirement of additional geometrically reducible fibers is part of the reason why Wittenberg's method, in its current form, does not apply to all degree 4 del Pezzo surfaces with $\mathrm{Br}\, X/\operatorname{im} \mathrm{Br}\, k \cong \mathbb{Z}/2\mathbb{Z}$.

Computing Brauer classes. The proof of Theorem 1.1 is constructive, and therefore gives an algorithm for computing explicit Brauer classes of X. In contrast to the existing algorithm developed by Bright, Bruin, Flynn, and Logan [BBFL07], our method does not proceed by computing the Galois action on the set of exceptional curves of X. Our algorithm is easily explained and quite fast in practice; it requires only:

(i) Computing determinants of 5×5 and 4×4 symmetric matrices,
(ii) Factoring a degree 5 polynomial,
(iii) Finding a rational point (which is known to exist) on a quadric surface, and finally
(iv) Taking partial derivatives of polynomials.

See §4.1 for details.

Related work.

(1) Colliot-Thélène, Harari, and Skorobogatov have studied a similar question: they proved a criterion [CTHS03, Prop. 2.5] for determining when $\mathrm{Br}\, X \subseteq f^* (\mathrm{Br}\, \mathbf{k}(\mathbb{P}^1))$ for $f \colon X \to \mathbb{P}^1$ a normic bundle, and gave several concrete examples of bundles satisfying their criterion [CTHS03, Cor. 2.6]. The salient difference between this article and theirs is that they work with an already specified morphism f.

(2) Any element in $\mathrm{Br}\, X$ of the form $(L/k, f)$, where L/k is a cyclic extension and $f \in k(X)^\times$, is plainly vertical for the map $X \dashrightarrow \mathbb{P}^1$, $x \mapsto f(x)$. Swinnerton-Dyer showed that all nonconstant Brauer classes on a del Pezzo surface of degree 4 are of this form [SD99, Ex. 3]. However, his functions $f \in k(X)^\times$ are ratios of polynomials whose degrees are not bounded by the construction. Moreover, the computation of these functions, and hence the computation of the map $X \dashrightarrow \mathbb{P}^1$, generally requires an explicit Galois descent of divisor classes.

Outline. We fix notation in §2.1. In §2.3 we study the rank 4 quadrics in the pencil associated to a del Pezzo surface X of degree 4, and use [BBFL07] to give a convenient set of generators for the geometric Picard group of X. In §2.4, we use [KST89] to describe the Galois action on these generators.

The heart of the paper is §3. In it, we construct explicit Brauer classes (§3.1) and use them to prove an explicit version of Theorem 1.1 for all fields of characteristic not 2 (§§3.2,3.3). In §4, we discuss computational applications of the results from §3: in §4.1, we describe an algorithm to compute nonconstant Brauer classes of X, and in §4.2 we show how to construct parameter spaces of degree 4 del Pezzo surfaces with a nonconstant Brauer class.

In §5 we restrict our attention to the case where the base field is a global field of characteristic not 2, and prove Theorem 1.1 and Corollary 1.3. In addition, in §5.3, we prove that the evaluation maps $\mathrm{ev}_{\mathcal{A}}$ for $\mathcal{A} \in \mathrm{Br}\, X$ are constant for a large set of places of k. In §5.4 we show that the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one for certain del Pezzo surfaces of degree 4 over number fields, assuming Schinzel's hypothesis and finiteness of Tate-Shafarevich groups. Finally, in §6, we discuss a partial improvement to Theorem 1.1 in the case where $\#(\mathrm{Br}\, X/\mathrm{im}\, \mathrm{Br}\, k) = 4$.

Acknowledgements

2. The Picard group of del Pezzo surfaces of degree 4

2.1. **Notation.** We fix much of the notation that will remain in force throughout the paper. Let k be a field of characteristic not 2, and fix a separable closure $\overline{k}$ of k. Let G_k denote the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For any homogeneous polynomial F in five variables, we write $V(F)$ for the corresponding variety in $\mathbb{P}^4$. For any smooth projective geometrically integral k-variety, we use $\sim$ to denote linear equivalence of divisors over $\overline{k}$.

Let X be a del Pezzo surface of degree 4 over a field k of characteristic not 2. By embedding it anticanonically, we will view X as a smooth complete intersection of two quadrics, Q and $\widetilde{Q}$, in $\mathbb{P}^4$. The forms Q and $\widetilde{Q}$ define a pencil $\{\lambda Q + \mu \widetilde{Q} : [\lambda : \mu] \in \mathbb{P}^1\}$ with five degenerate geometric fibers, each a rank 4 quadric [Wit07, Prop. 3.26].

We write $\mathscr{S} \subseteq \mathbb{P}^1$ for the degree 5 subscheme defining the degeneracy locus of the pencil. We denote the $\overline{k}$-points of $\mathscr{S}$ by $t_0, \ldots, t_4$; let $k(t_i)$ be the smallest subfield of $\overline{k}$ that contains t_i, and let Q_i denote the rank 4 quadric associated to t_i, which is defined over $k(t_i)$. We write ε_{t_i} for the discriminant of the smooth rank 4 quadric obtained by restricting $V(Q_i)$ to a hyperplane $H \subseteq \mathbb{P}^4$ not containing the vertex of $V(Q_i)$. Throughout, we consider ε_{t_i} as an element of $k(t_i)^{\times}/k(t_i)^{\times 2}$; as such, it does not depend on the choice of H [Wit07, §3.4.1].

Similarly, for any $T \in \mathscr{S}$, we write $\kappa(T)$ for the residue field of T, Q_T for the corresponding rank 4 quadric, and $\varepsilon_T \in \kappa(T)^\times/\kappa(T)^{\times 2}$ for the discriminant of the restriction of $V(Q_T)$ to a suitable hyperplane as above.

As in the introduction, the Brauer group of X is $\operatorname{Br} X := \operatorname{H}^2_{\text{et}}(X, \mathbb{G}_m)$. We write $\operatorname{Br}_0 X :=$ $\operatorname{im} \operatorname{Br} k \to \operatorname{Br} X$ for the subgroup of constant algebras. For a dominant map $f \colon X \dashrightarrow Y$, we define the vertical Brauer group of X with respect to f as

$$\operatorname{Br}^{(f)}_{\text{vert}} X := \operatorname{Br} X \cap f^* \left(\operatorname{Br} \mathbf{k}(Y) \right).$$

More generally, a map $f \colon X \dashrightarrow Y$ gives rise to pullback map $f^* \colon \operatorname{Br} \mathcal{O}_{Y,f(\eta)} \to \operatorname{Br} \mathbf{k}(X)$, where $\mathcal{O}_{Y,f(\eta)}$ is the local ring of Y at the image of the generic point η of X. In this case, we define

$$\operatorname{Br}^{(f)}_{\text{vert}} X = \operatorname{Br} X \cap f^* \left(\operatorname{Br} \mathcal{O}_{Y,f(\eta)} \right).$$

2.2. Linear spaces on rank 4 quadrics.

Lemma 2.1. *Assume that $V(Q_T)$ has a smooth $\kappa(T)$-point P. Then there exists a set of linearly independent $\kappa(T)$-linear forms $\ell_1, \ell_2, \ell_3,$ and ℓ_4 such that $\ell_2(P) = \ell_3(P) = \ell_4(P) = 0$ and*

$$cQ_T = \ell_1\ell_2 - \ell_3^2 + \varepsilon_T\ell_4^2,$$

for some $c \in \kappa(T)^\times$. In particular, if H is a hyperplane tangent to $V(Q_T)$ at P, then $X \cap H$ is geometrically reducible and $V(Q_T) \cap H = L \cup L'$ for some planes L, L' defined over $\kappa(T) \left(\sqrt{\varepsilon_T} \right)$.

Proof. Consider the quadratic form Q_T as a quadric in four variables. We may view P as an isotropic vector of the corresponding quadratic module. This module contains a hyperbolic plane U containing P, spanned by $\kappa(T)$-linear forms ℓ_1 and ℓ_2. Now diagonalize the restriction of Q_T to the 2-dimensional orthogonal complement of U. Scaling Q_T by an appropriate nonzero constant c and comparing discriminants, we may assume that $cQ_T|_{U^\perp} \sim -\ell_3^2 + \varepsilon_T\ell_4^2$ for some $\kappa(T)$-linear forms ℓ_3 and ℓ_4. $\square$

2.3. Generators for the geometric Picard group. For $i \in \{0, \ldots, 4\}$, write $W_i := V(Q_i)$ and let k_i be a finite extension of $k(t_i)$ such that

$$W_i \text{ has a smooth } k_i\text{-point, } \quad \text{and} \quad [k_i(\sqrt{\varepsilon_{t_i}}) : k_i] = [k(t_i)(\sqrt{\varepsilon_{t_i}}) : k(t_i)].$$

For any smooth point $P_i \in W_i(k_i)$, let H_{P_i} be the hyperplane tangent to W_i at P_i. By Lemma 2.1, we have $W_i \cap H_{P_i} = L_{P_i} \cup L'_{P_i}$, where L_{P_i} and L'_{P_i} are planes defined over $k_i \left(\sqrt{\varepsilon_{t_i}} \right)$.

Define

$$C_{P_i} = X \cap L_{P_i} \quad \text{and} \quad C'_{P_i} = X \cap L'_{P_i}.$$

As in [BBFL07, Theorem 4], note that

$$C_{P_i} = X \cap L_{P_i} = W \cap W_i \cap L_{P_i} = W \cap L_{P_i},$$

where W is a quadric hypersurface associated to a smooth quadric in the pencil defined by Q and $\widetilde{Q}$. Since $W \cap L_{P_i}$ is the intersection of a smooth quadric 3-fold with a plane, the curve C_{P_i} is a conic; the same is true for C'_{P_i}. By construction, these conics are defined over the field $k_i \left(\sqrt{\varepsilon_{t_i}} \right)$.

A different choice of smooth point $\tilde{P}_i \in W_i(k_i)$ gives rise to two different planes $L_{\tilde{P}_i}$ and $L'_{\tilde{P}_i}$, and hence two different conics $C_{\tilde{P}_i}$ and $C'_{\tilde{P}_i}$. Since the planes $L_{\tilde{P}_i}$, $L'_{\tilde{P}_i}$, L_{P_i} and L'_{P_i} are contained in two pencils, without loss of generality, we may assume that $C_{\tilde{P}_i} \sim C_{P_i}$ and $C'_{\tilde{P}_i} \sim C'_{P_i}$. In particular, for any element $\sigma \in G_{k(t_i, \sqrt{\varepsilon_{t_i}})}$, we have $C_{\sigma(P_i)} = \sigma(C_{P_i})$. Henceforth, we write C_i and C'_i for the classes in $\operatorname{Pic}\overline{X}$ of C_{P_i} and $C_{P'_i}$ respectively. This discussion shows that C_i and C'_i are defined over $k(t_i, \sqrt{\varepsilon_{t_i}})$.

Proposition 2.2. *After possibly interchanging C_i and C'_i for some indices i, we may assume that the group $\operatorname{Pic}(\overline{X}) \cong \mathbb{Z}^6$ is freely generated by the classes of the following divisors*

$$\frac{1}{2}(H + C_0 + C_1 + C_2 + C_3 + C_4), \ C_0, \ C_1, \ C_2, \ C_3, \text{ and } C_4, \tag{2.1}$$

where H is a hyperplane section of X, and the C_i are conics as above.

Proof. Recall that, by the definition of a del Pezzo surface, $\overline{X}$ is $\overline{k}$-isomorphic to the blow-up of $\mathbb{P}^2$ at five points $\{p_0, \dots, p_4\}$, no three of which are colinear. For $0 \le i \le 4$ write E_i for the exceptional divisor corresponding to p_i under the blow-up map, and let L be the strict transform of a line in $\mathbb{P}^2$ that does not pass through any of the points p_i.

Since X is anticanonically embedded, we have $H \sim -K_X \sim 3L - \sum E_i$. By [BBFL07, Theorems 2 and 4], after possibly interchanging C_i and C'_i, we may assume that $C_i \sim L - E_i$ and $C'_i \sim -K_X - C_i$. Thus

$$\frac{1}{2}(H + C_0 + C_1 + \cdots C_4) \sim \frac{1}{2}(3L - \sum E_i + \sum(L - E_i)) \sim 4L - \sum E_i \in \operatorname{Pic}\overline{X}.$$

Using the above expressions for C_i and C'_i in terms of L and E_i, we compute that

$$C_i^2 = (C'_i)^2 = 0, \quad C_i \cdot H = C_i \cdot C'_i = 2, \text{ and } C_i \cdot C_j = C'_i \cdot C'_j = C_i \cdot C'_j = 1 \text{ for } i \ne j.$$

These intersection numbers imply that the divisor classes (2.1) span a rank 6 sublattice of $\operatorname{Pic}(\overline{X})$. Furthermore, this sublattice is unimodular, so it must be equal to all of $\operatorname{Pic}(\overline{X})$. $\square$

2.4. **Galois action on** $\operatorname{Pic}(\overline{X})$. Let Γ be a graph on ten vertices indexed by the conics $C_0, C'_0, \dots, C_4, C'_4$ whose only edges join C_i to C'_i for $i = 0, \dots, 4$. Let $c_i \in \operatorname{Aut}(\Gamma)$ be the graph automorphism that exchanges C_i with C'_i and leaves every other vertex fixed. The group $\operatorname{Aut}(\Gamma)$ is a semi-direct product

$$(\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_5,$$

where the S_5 factor permutes the set of pairs $\{\{C_i, C'_i\} : 0 \le i \le 4\}$, and c_i generates one of the $\mathbb{Z}/2\mathbb{Z}$-factors. By the discussion in [KST89, pp. 8–10], there is a natural embedding of index 2

$$O(K_X^\perp) \hookrightarrow \operatorname{Aut}(\Gamma).$$

An element in $\operatorname{Aut}(\Gamma)$ is in the image of this map if and only if it is a product of an even number of c_i's with an element of S_5.

The group G_k acts on $\operatorname{Pic}(\overline{X})$, and it preserves the intersection pairing, as well as K_X. The action therefore factors through the group $O(K_X^\perp)$, which we consider in turn as a subgroup of $\operatorname{Aut}(\Gamma)$, as above. Projection onto the S_5-factor gives the usual action of G_k on the set $\mathscr{S}(\overline{k}) = \{t_0, \dots, t_4\}$.

For any $T \in \mathscr{S}$, let Γ_T be the graph with $2 \deg(T)$ vertices indexed by C_i and C_i' for $t_i \in T(\overline{k})$, whose only edges join C_i and C_i' for $t_i \in T(\overline{k})$. Note that Γ_T is a subgraph of Γ, and that Γ is the union of Γ_T over all $T \in \mathscr{S}$. There is a natural injective map

$$\prod_{T \in \mathscr{S}} \operatorname{Aut}(\Gamma_T) \hookrightarrow \operatorname{Aut}(\Gamma),$$

which we use to identify the domain with its image in $\operatorname{Aut}(\Gamma)$.

Proposition 2.3. *The action of the group G_k on $\operatorname{Pic}(\overline{X})$ induces an action on Γ that factors through*

$$\left(\prod_{T \in \mathscr{S}} \operatorname{Aut}(\Gamma_T) \cap O(K_X^\perp) \right) \subseteq \operatorname{Aut}(\Gamma).$$

Moreover, G_k acts transitively on the set $\{C_i, C_i' : t_i \in T(\overline{k})\}$ if and only if $\varepsilon_T \notin \kappa(T)^{\times 2}$.

Proof. The first part of the proposition follows from the above discussion. For the last claim, it suffices to note that for $t_i \in T(\overline{k})$, the pair of divisor classes $\{C_i, C_i'\}$ is defined over $k(t_i)$, whilst the constituent classes of the pair are defined over $k\left(t_i, \sqrt{\varepsilon_{t_i}}\right)$. $\qquad\square$

3. All Brauer elements are vertical

We keep the notation from the previous section. Throughout, we assume that for all $T \in \mathscr{S}$, the quadric hypersurface $V(Q_T)$ has a smooth $\kappa(T)$-point.

3.1. The construction. Assume that there exists a k-rational subscheme $\mathscr{T} \subseteq \mathscr{S}$ such that

$$\deg(\mathscr{T}) = 2, \quad \prod_{T \in \mathscr{T}} \operatorname{N}_{\kappa(T)/k}(\varepsilon_T) \in k^{\times 2}, \quad \text{and } \varepsilon_T \notin \kappa(T)^{\times 2} \text{ for all } T \in \mathscr{T}. \qquad (\star)$$

Lemma 3.1. *If $\mathscr{T}$ satisfies $(\star)$, then ε_T lies in the image of $k^\times/k^{\times 2} \to \kappa(T)^\times/\kappa(T)^{\times 2}$ for all $T \in \mathscr{T}$.*

Proof. If $\mathscr{T}$ contains a degree 1 point, then the claim is immediate as $\kappa(T) = k$ for all $T \in \mathscr{T}$. Assume that $T \in \mathscr{T}$ is a degree 2 point, and consider the exact sequence of Galois modules

$$0 \to \mathbb{Z}/2 \to \operatorname{Ind}_{\{\mathrm{id}\}}^{\operatorname{Gal}(\kappa(T)/k)} (\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0.$$

The long exact sequence from cohomology yields the exact sequence

$$k^\times/k^{\times 2} \to \kappa(T)^\times/\kappa(T)^{\times 2} \xrightarrow{\operatorname{N}_{\kappa(T)/k}} k^\times/k^{\times 2}$$

which completes the proof. $\qquad\square$

If $\mathscr{T}$ satisfies $(\star)$, then the lemma allows us to assume that $\varepsilon_T \in k^\times$ for all $T \in \mathscr{T}$. Define $L_{\mathscr{T}} = k(\sqrt{\varepsilon_T})$ for some $T \in \mathscr{T}$; this extension is a quadratic extension, independent of the choice of T.

By assumption, for all $T \in \mathscr{T}$, $V(Q_T)$ has a smooth $\kappa(T)$-point. Let ℓ_T be a $\kappa(T)$-linear form such that the associated hyperplane is tangent to $V(Q_T)$ at a smooth point.

Lemma 3.2. *Assume that $\mathscr{T}$ satisfies $(\star)$, and let ℓ be any k-linear form. Then the cyclic algebra*

$$\mathcal{A}_\mathscr{T} := \left(L_\mathscr{T}/k, \ell^{-2} \prod_{T \in \mathscr{T}} \mathrm{N}_{\kappa(T)/k}(\ell_T) \right)$$

is unramified and thus is in the image of $\mathrm{Br}\, X \to \mathrm{Br}\, \mathbf{k}(X)$.

Proof. By the purity theorem [Gro68, §§6,7], the class of a cyclic algebra $(L/k, f)$ is in the image of $\mathrm{Br}\, X \to \mathrm{Br}\, \mathbf{k}(X)$ if and only if for every prime divisor $V \subseteq X$ we have $L \subseteq \kappa(V)$ whenever $\mathrm{ord}_V(f) \equiv 1 \bmod 2$. Therefore, we must show that $L_\mathscr{T} \subseteq \kappa(C_T \cup C'_T)$ for all $T \in \mathscr{T}$. This containment holds because C_T and C'_T are conjugate over $L_\mathscr{T}$. $\qquad\square$

3.2. **Nontriviality of $\mathcal{A}_\mathscr{T}$.** We would like to determine when $\mathcal{A}_\mathscr{T}$ is nontrivial, i.e., when $\mathcal{A}_\mathscr{T} \notin \mathrm{Br}_0 X$. Let L be a cyclic extension of k, and let $f \in \mathbf{k}(X)^\times$. Define

$$\mathrm{Br}_{\mathrm{cyc}}(X, L) = \left\{ \begin{array}{l} \text{classes } [(L/k, f)] \text{ in the image of the} \\ \text{map } \mathrm{Br}\, X / \mathrm{Br}_0 X \to \mathrm{Br}\, \mathbf{k}(X) / \mathrm{Br}_0 X \end{array} \right\}$$

Fix a generator σ of $\mathrm{Gal}(L_\mathscr{T}/k)$. We view $\mathrm{N}_{L_\mathscr{T}/k}$ and $1 - \sigma$ as endomorphisms of $\mathrm{Pic}(X_L)$, and we consider the image of $\mathcal{A}_\mathscr{T}$ under the composition of the following maps

$$\mathrm{Br}_{\mathrm{cyc}}(X, L_\mathscr{T}) \xrightarrow{\sim} \frac{\ker \mathrm{N}_{L_\mathscr{T}/k}}{\mathrm{im}(1 - \sigma)} \hookrightarrow \mathrm{H}^1(G_k, \mathrm{Pic}(\overline{X}))[2] \xrightarrow{\sim} \frac{\left(\mathrm{Pic}\,\overline{X} / 2\,\mathrm{Pic}\,\overline{X} \right)^{G_k}}{\left((\mathrm{Pic}\,\overline{X})^{G_k} / 2(\mathrm{Pic}\,\overline{X})^{G_k} \right)}. \quad (3.1)$$

The first two maps are described in [VA08, Thm 3.3], and the last map is obtained by considering the long exact sequence in cohomology associated to the short exact sequence $0 \to \mathrm{Pic}\,\overline{X} \xrightarrow{\times 2} \mathrm{Pic}\,\overline{X} \to \left(\mathrm{Pic}\,\overline{X} / 2\,\mathrm{Pic}\,\overline{X} \right) \to 0$. (In [VA08, §3] k is assumed to be a number field; however, this hypothesis is not necessary for the result we cite.)

By [VA08, Thm 3.3], a degree 2 cyclic algebra $(L/k, f)$ maps to a cocycle in $\mathrm{H}^1(G_k, \mathrm{Pic}\,\overline{X})$ that sends G_L to the identity, and that sends any element outside of G_L to a divisor D such that $\mathrm{div}(f) = \sum_{\sigma \in \mathrm{Gal}(L/k)} \sigma(D)$. Therefore, the cyclic algebra $\mathcal{A}_\mathscr{T}$ maps to the cocycle

$$\sigma \mapsto \begin{cases} -H + \sum_{t_i \in \mathscr{T}(\overline{k})} C_i & \text{if } \sigma \notin G_{L_\mathscr{T}}, \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

in $\mathrm{H}^1(G_k, \mathrm{Pic}\,\overline{X})$. Since $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i$ is fixed by every element of $G_{L_\mathscr{T}}$ and

$$\frac{1}{2} \left(\sum_{t_i \in \mathscr{T}(\overline{k})} C_i - \sigma \left(\sum_{t_i \in \mathscr{T}(\overline{k})} C_i \right) \right) = \frac{1}{2} \left(\sum_{t_i \in \mathscr{T}(\overline{k})} C_i - \sum_{t_i \in \mathscr{T}(\overline{k})} C'_i \right) = -H + \sum_{t_i \in \mathscr{T}(\overline{k})} C_i,$$

for all $\sigma \in G_k \setminus G_{L_\mathscr{T}}$, the cocycle (3.2) maps to $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i$ under the last map in (3.1). Thus $\mathcal{A}_\mathscr{T}$ is trivial if and only if $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i \in 2\,\mathrm{Pic}\,\overline{X} + (\mathrm{Pic}\,\overline{X})^{G_k}$.

Proposition 3.3. *If $\mathscr{T}$ is a k-subscheme of $\mathscr{S}$ that satisfies $(\star)$, then $\mathcal{A}_\mathscr{T} \notin \mathrm{Br}_0(X)$ if and only if there exists $T \in \mathscr{S} \setminus \mathscr{T}$ such that $\varepsilon_T \notin \kappa(T)^{\times 2}$.*

Proof. As noted above, $\mathcal{A}_\mathscr{T}$ is trivial if and only if $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i \in 2\,\mathrm{Pic}\,\overline{X} + (\mathrm{Pic}\,\overline{X})^{G_k}$. By Proposition 2.2 the Picard group is freely generated by

$$\frac{1}{2} \left(H + C_0 + C_1 + C_2 + C_3 + C_4 \right), \ C_0, \ C_1, \ C_2, \ C_3, \text{ and } C_4.$$

Since, by $(\star)$, $\varepsilon_T \notin k(T)^{\times 2}$ for all $T \in \mathscr{T}$, Proposition 2.3 implies that $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i$ is not fixed by Galois. Therefore, $\sum_{t_i \in \mathscr{T}(\overline{k})} C_i \in 2\operatorname{Pic} \overline{X} + (\operatorname{Pic} \overline{X})^{G_k}$ if and only if there is some choice of sign such that

$$\sum_{t_i \in (\mathscr{S} \setminus \mathscr{T})(\overline{k})} \pm C_i \in (\operatorname{Pic} \overline{X})^{G_k}.$$

Assume that there exists $T \in \mathscr{S} \setminus \mathscr{T}$ with $\varepsilon_T \notin \kappa(T)^{\times 2}$. Then, by Proposition 2.3, the absolute Galois group acts transitively on $\{C_i, C_i' : t_i \in T(\overline{k})\}$. Thus, for some $t_j \in (\mathscr{S} \setminus \mathscr{T})(\overline{k})$, there exists an element $\sigma \in G_k$ such that

$$\sigma \left(\pm C_j + \sum_{t_i \in (\mathscr{S} \setminus \mathscr{T})(\overline{k}), i \neq j} \pm C_i \right) = \mp C_j + D,$$

where D is a divisor that does not involve C_j. Therefore, for any choice of signs,

$$\sum_{t_i \in (\mathscr{S} \setminus \mathscr{T})(\overline{k})} \pm C_i \notin (\operatorname{Pic} \overline{X})^{\sigma} \supseteq (\operatorname{Pic} \overline{X})^{G_k},$$

and so $\mathcal{A}_{\mathscr{T}}$ is nontrivial in $\operatorname{Br} X / \operatorname{Br}_0 X$.

Now assume that for all $T \in \mathscr{S} \setminus \mathscr{T}$, $\varepsilon_T \in \kappa(T)^{\times 2}$. By Proposition 2.3, G_k does not act transitively on $\{C_i, C_i' : t_i \in T(\overline{k})\}$. Since G_k does act transitively on $\{t_i : t_i \in T(\overline{k})\}$, the set $\{C_i, C_i' : t_i \in T(\overline{k})\}$ breaks up into two Galois invariant sets of equal size such that for all $t_i \in T(\overline{k})$, C_i and C_i' are not contained in the same set.

Without loss of generality, we may assume that $(\mathscr{S} \setminus \mathscr{T})(\overline{k}) = \{t_0, t_1, t_2\}$. The previous discussion shows that, possibly after renumbering, at least one of

$$\{C_0, C_1, C_2\} \quad \text{or} \quad \{C_0, C_1, C_2'\}$$

are Galois invariant. If the first set is Galois invariant, then $C_0 + C_1 + C_2 \in (\operatorname{Pic} \overline{X})^{G_k}$ and if the latter is Galois invariant, then $C_0 + C_1 - C_2 \in (\operatorname{Pic} \overline{X})^{G_k}$. In either case, this shows that $\mathcal{A}_{\mathscr{T}}$ is in $\operatorname{Br}_0 X$. $\qquad\square$

3.3. Main result.

Theorem 3.4. *If $\operatorname{Br} X \neq \operatorname{Br}_0 X$ then there exists $\mathscr{T} \subseteq \mathscr{S}$ satisfying $(\star)$ and $T \in \mathscr{S} \setminus \mathscr{T}$ such that $\varepsilon_T \notin \kappa(T)^{\times 2}$. Furthermore, if $\#(\operatorname{Br} X / \operatorname{Br}_0 X) = 4$ then there exist $T_0, T_1, T_2 \in \mathscr{S}(k)$ such that every pair satisfies $(\star)$.*

If, for all $T \in \mathscr{S}$ of degree at most 2, the quadric hypersurface $V(Q_T)$ has a smooth $\kappa(T)$-point, then the converses of the above two statements hold. Moreover, any nontrivial element in $\operatorname{Br} X / \operatorname{Br}_0 X$ is of the form $\mathcal{A}_{\mathscr{T}}$ for some $\mathscr{T} \subseteq \mathscr{S}$, and if $\#(\operatorname{Br} X / \operatorname{Br}_0 X) = 4$, then $\operatorname{Br} X / \operatorname{Br}_0 X = \{\operatorname{id}, \mathcal{A}_{\{T_0, T_1\}}, \mathcal{A}_{\{T_0, T_2\}}, \mathcal{A}_{\{T_1, T_2\}}\}$.

Proof. From the Hochschild-Serre spectral sequence, we have an injection

$$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X} \hookrightarrow \operatorname{H}^1(G_k, \operatorname{Pic} \overline{X});$$

a priori, this map need not be surjective for an arbitrary field k of characteristic not 2. Using Proposition 2.3 and a case by case analysis, one can compute that $\# \operatorname{H}^1(G_k, \operatorname{Pic} \overline{X}) \mid 4$, and that if $\# \operatorname{H}^1(G_k, \operatorname{Pic} \overline{X}) = 2$, then there exists $\mathscr{T} \subset \mathscr{S}$ satisfying $(\star)$, and a $T' \in \mathscr{S} \setminus \mathscr{T}$ such that $\varepsilon_{T'} \notin \kappa(T')^{\times 2}$. By assumption, $V(Q_T)$ has a smooth $\kappa(T)$-point for all $T \in$

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$\mathscr{T}$, so $\mathcal{A}_{\mathscr{T}}$ exists, and by Proposition 3.3 it is a nontrivial element of $\operatorname{Br} X / \operatorname{Br}_0 X$. Since $\#(\operatorname{Br} X / \operatorname{Br}_0 X) \leq 2$, $\mathcal{A}_{\mathscr{T}}$ must be the unique nontrivial element.

Furthermore, we compute that if $\# \operatorname{H}^1(G_k, \operatorname{Pic} \overline{X}) = 4$, then there exist T_0, T_1, and T_2 in $\mathscr{S}(k)$ such that every pair $\{T_i, T_j\}$ satisfies ($\star$). As above, we see that

$$\mathcal{A}_{\{T_0, T_1\}}, \quad \mathcal{A}_{\{T_0, T_2\}}, \quad \text{and} \quad \mathcal{A}_{\{T_1, T_2\}}$$

exist and are nontrivial elements of $\operatorname{Br} X / \operatorname{Br}_0 X$. Moreover, from the construction it is clear that $\mathcal{A}_{\{T_1, T_2\}} = \mathcal{A}_{\{T_0, T_1\}} + \mathcal{A}_{\{T_0, T_2\}}$, so $\#(\operatorname{Br} X / \operatorname{Br}_0 X) = 4$ and these elements are the three nontrivial elements in $\operatorname{Br} X / \operatorname{Br}_0 X$. A Magma [BCP97] script to verify these claims can be found in the arXiv distribution of this article. Alternatively, the industrious reader may carry out these computations by hand using arguments similar to those in the proof of Proposition 3.3. $\qquad \square$

When k is a number field, work of Wittenberg enables us to give an alternative proof [Wit07].

Proof of Theorem 3.4 assuming that k is a number field. Let $\mathscr{S}_1 \subseteq \mathscr{S}$ be the set of $T \in \mathscr{S}$ such that $\varepsilon_T \notin \kappa(T)^{\times 2}$, and let $\mathscr{S}_0 \subseteq \mathscr{S}_1$ be a minimal generating set for

$$\langle \operatorname{N}_{\kappa(T)/k}(\varepsilon_T) : T \in \mathscr{S} \rangle \subseteq k^\times / k^{\times 2}.$$

By [Wit07, Thm. 3.37], $\#(\operatorname{Br} X / \operatorname{Br}_0 X) = 2^{\max(0, n-d-1)}$ where $d = \# \mathscr{S}_0$ and $n = \# \mathscr{S}_1$, so, hereafter, we assume $n - d \geq 2$. Throughout the proof, we will often rely on [Wit07, Prop. 3.39], which states that

$$\prod_{T \in \mathscr{S}} \operatorname{N}_{\kappa(T)/k}(\varepsilon_T) \in k^{\times 2}. \tag{3.3}$$

Also, when discussing multiplicative independence of elements of the form $\operatorname{N}_{\kappa(T)/k}(\varepsilon_T)$ below, we are implicitly working in $k^\times / k^{\times 2}$.

We will show that if $n - d = 2$, then there exists $\mathscr{T} \subset \mathscr{S}$ satisfying ($\star$), and for which $\mathscr{S}_1 \setminus \mathscr{T} \neq \emptyset$. By Proposition 3.3, this implies that $\mathcal{A}_{\mathscr{T}}$ is the unique nontrivial element of $\operatorname{Br} X / \operatorname{Br}_0 X$. We will also show that $n - d \leq 3$ and that if $n - d = 3$, then there exist three points T_0, T_1, and T_2 of $\mathscr{S}(k)$ such that every pair satisfies ($\star$). Then, by Proposition 3.3, $\mathcal{A}_{\{T_0, T_1\}}, \mathcal{A}_{\{T_0, T_2\}}, \mathcal{A}_{\{T_1, T_2\}}$ are nontrivial elements of $\operatorname{Br} X / \operatorname{Br}_0 X$. Moreover, from the construction it is clear that $\mathcal{A}_{\{T_1, T_2\}} = \mathcal{A}_{\{T_0, T_1\}} + \mathcal{A}_{\{T_0, T_2\}}$, so these elements are the three nontrivial elements in $\operatorname{Br} X / \operatorname{Br}_0 X$. These two claims will complete the proof of the theorem.

Consider the case when $n = 2$. Then we must have $d = 0$, which means that $\mathscr{S}_1$ must consist entirely of points of degree > 1. Since $\mathscr{S}$ has total degree 5, there is a degree 2 point $T \in \mathscr{S}$. Take $\mathscr{T} := \{T\}$; note that it satisfies ($\star$), and that $\mathscr{S}_1 \setminus \mathscr{T} \neq \emptyset$.

Henceforth, we assume that $n > 2$. This means that for any degree 2 subscheme $\mathscr{T} \subset \mathscr{S}_1$, we have $\mathscr{S}_1 \setminus \mathscr{T} \neq \emptyset$. Suppose that $n = 3$. Then $\mathscr{S}$ contains at least $(\# \mathscr{S} - 2)$ k-points. This implies that there is a $T_0 \in \mathscr{S}(k)$ with $\varepsilon(T_0) \notin \kappa(T_0)^{\times 2}$, which in turn shows that $d = 1$, because $n - d \geq 2$. Without loss of generality, take $\mathscr{S}_0 = \{T_0\}$. Then either there is a $T_1 \in \mathscr{S}(k)$ with $\varepsilon_{T_1} \notin \kappa(T_1)^{\times 2}$, in which case the set $\mathscr{T} := \{T_0, T_1\}$ satisfies ($\star$), or the degree sequence of $\mathscr{S}$ is $(1, 2, 2)$. In the latter case, the relation (3.3) implies that exactly one of the two degree 2 points $T \in \mathscr{S}$ satisfies $\operatorname{N}_{\kappa(T)/k}(\varepsilon_T) \in k^{\times 2}$. Then $\mathscr{T} := \{T\}$ satisfies ($\star$).

Next, assume that $n = 4$ and $d \leq 2$. Then there exist at least three degree 1 points T_0, T_1 and T_2 in $\mathscr{S}_1$, so, in particular, $d > 0$. If $d = 1$, then every order 2 subset of $\{T_0, T_1, T_2\}$ satisfies ($\star$). If $d = 2$, then either every order 2 subset of $\{T_0, T_1, T_2\}$ is multiplicatively independent, in which case $\mathscr{T} := \mathscr{S} \setminus \{T_0, T_1, T_2\}$ satisfies ($\star$), or there is exactly one order

2 subset of $\{T_0, T_1, T_2\}$ that is multiplicatively dependent. In the latter case, this unique subset satisfies $(\star)$.

It remains to consider the case when $n = 5$ and $d \le 3$. By (3.3), we have $d \ge 2$. Write $\mathscr{S} = \{t_0, t_1, t_2, t_3, t_4\}$. Without loss of generality, assume that ε_{t_0} and ε_{t_1} are multiplicatively independent. If $d = 2$, then, up to renumbering, we have $\varepsilon_{t_4} = \varepsilon_{t_0}\varepsilon_{t_1}$, and either

$$\varepsilon_{t_2} = \varepsilon_{t_3} = \varepsilon_{t_4} \quad \text{or} \quad \varepsilon_{t_0} = \varepsilon_{t_2} = \varepsilon_{t_3}.$$

In either case, we obtain our desired result. If $d = 3$, then we may assume that ε_{t_0}, ε_{t_1} and ε_{t_2} are multiplicatively independent, that $\varepsilon_{t_3} = \varepsilon_{t_0}\varepsilon_{t_1}$, and that $\varepsilon_{t_2} = \varepsilon_{t_4}$. Then $\mathscr{T} := \{t_2, t_4\}$ satisfies $(\star)$. $\qquad\square$

Corollary 3.5. *Assume that for all $T \in \mathscr{S}(k)$, the quadric hypersurface $V(Q_T)$ has a smooth k-point. Then every element of $\operatorname{Br} X / \operatorname{Br}_0 X$ is vertical for a genus 1 fibration $X \dashrightarrow \mathbb{P}^1$ that has at most 2 geometrically reducible fibers.*

Proof. Let $\mathcal{A}$ be a nontrivial element of $\operatorname{Br} X / \operatorname{Br}_0 X$. By Theorem 3.4, $\mathcal{A} = \mathcal{A}_{\mathscr{T}}$ for some $\mathscr{T} \subseteq \mathscr{S}$ satisfying $(\star)$. If $\mathscr{T} = \{T_0, T_1\}$, let ℓ_i be a k-linear form tangent to $V(Q_{T_i})$ at a smooth point for $i = 0, 1$. Otherwise, $\mathscr{T}$ consists of a degree 2 point T; in this case let d be the discriminant of the residue field of T, and let ℓ_0 and ℓ_1 be k-linear forms such that $\ell_0 + \sqrt{d}\ell_1$ is tangent to $V(Q_T)$ at a smooth point.

Define a map $f \colon X \dashrightarrow \mathbb{P}^1$ by $x \mapsto [\ell_0(x) : \ell_1(x)]$. Since $\mathcal{A}_{\mathscr{T}}$ is equal to either

$$\left(L_{\mathscr{T}}/k, \frac{\ell_0}{\ell_1} \right) \quad \text{or} \quad \left(L_{\mathscr{T}}/k, \frac{\ell_0^2 - d\ell_1^2}{\ell_1^2} \right),$$

it is clear that $\mathcal{A}_{\mathscr{T}} \in \operatorname{Br}^{(f)}_{\text{vert}} X$. Moreover, every fiber of f is an intersection of a hyperplane with X, i.e., a genus 1 curve.

It remains to prove that f has at most 2 reducible fibers. Since the $\kappa(T)$-points on $V(Q_T)$ are Zariski dense, it suffices to show that the locus of $\{\ell_0, \ell_1\}$ that give rise to a rational map with at least 3 geometrically reduced fibers is a proper closed subset of $\prod_{T \in \mathscr{T}} R_{\kappa(T)/k}(V(Q_T))$, where $R_{\kappa(T)/k}(-)$ denotes Weil restriction of scalars. Thus, we may pass to the separable closure $\overline{k}$.

First consider the case when $\#\mathscr{T} = 2$. After a change of variables, we may assume that $\{Q_t : t \in \mathscr{T}(\overline{k})\}$ equals $\{Q_1, Q_2\}$ where

$$Q_1 := x_0^2 - x_2^2 - x_3^2 + \gamma^2 x_4^2, \quad Q_2 := x_1^2 - \alpha^2 x_2^2 + \beta^2 x_3^2 - x_4^2,$$

for some $\alpha, \beta, \gamma \in \overline{k}^{\times}$. The locus of maps with at least 3 geometrically reduced fibers is closed; we must show that it is proper. Consider the linear forms

$$\ell_1 := x_0 - x_2 - \gamma x_3 + \gamma^2 x_4, \quad \ell_2 := x_1 - \alpha x_2 + \beta^2 \alpha^{-1} x_3 - \beta \alpha^{-1} x_4.$$

A computation shows that $V(\ell_1)$ and $V(\ell_2)$ are respectively tangent to $V(Q_1)$ and $V(Q_2)$ at smooth points, and that projection from $V(\ell_1, \ell_2)$ gives a map with at most 2 geometrically reduced fibers.[1]

Now consider the case when $\mathscr{T}$ consists of a single degree 2 point T. Then

$$\prod_{T \in \mathscr{T}} R_{\kappa(T)/k}(V(Q_T))(\overline{k}) = \left(V(Q_t)(\overline{k}) \right)^2$$

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[1] A `Magma` [BCP97] script to verify this claim and the similar claim below can be found in the `arXiv` distribution of this article.

for some $t \in T(\overline{k})$. After a change of variables, we may assume that $Q_t = Q_1$ above, and that Q_2 is some other quadric in the pencil. Consider the linear forms

$$\ell_1 := x_0 - x_2 - \gamma x_3 + \gamma^2 x_4, \quad \ell_2 := x_0 - \gamma x_2 - x_3 + \gamma^2 x_4.$$

A computation shows that $V(\ell_1)$ and $V(\ell_2)$ are both tangent to $V(Q_1)$ at smooth points, and that projection from $V(\ell_1, \ell_2)$ gives a map with at most 2 geometrically reduced fibers. This completes the proof. $\qquad\square$

Corollary 3.6. *Assume that for all $T \in \mathscr{S}$ of degree at most 2, the quadric hypersurface $V(Q_T)$ has a smooth $\kappa(T)$-point. If $\#(\mathrm{Br}\, X / \mathrm{Br}_0 X) = 4$, then every nontrivial element is cyclic for the same quadratic extension.*

Proof. It follows from the definition of $L_{\mathscr{T}}$ given in Lemma 3.2 that $L_{\{T_0, T_1\}} = L_{\{T_0, T_2\}} = L_{\{T_1, T_2\}}$. $\qquad\square$

4. Computational applications

4.1. **A practical algorithm for computing** $\mathrm{Br}\, X / \mathrm{Br}_0 X$**.** The construction given in §3.1, together with Theorem 3.4, yields a practical algorithm for the computation of $\mathrm{Br}\, X / \mathrm{Br}_0 X$, provided that for all $T \in \mathscr{S}$, the quadric hypersurface $V(Q_T)$ has a smooth $\kappa(T)$-point. (This hypothesis is satisfied, for example, whenever k is a global field of characteristic not 2 and $X(\mathbb{A}_k) \neq \emptyset$; see Lemma 5.1.) When k is a number field, Bright, Bruin, Flynn, and Logan have implemented a different algorithm to compute $\mathrm{Br}\, X / \mathrm{Br}_0 X$ [BBFL07]. Our algorithm has a simple implementation and in practice seems to be competitive with [BBFL07].

The algorithm takes as input two quadrics Q and $\widetilde{Q}$ with coefficients in k such that $X = V(Q) \cap V(\widetilde{Q})$, and gives as output a complete list of the elements of $\mathrm{Br}\, X / \mathrm{Br}_0 X$.

(1) Let M and $\widetilde{M}$ be the symmetric matrices associated to the defining quadrics Q and $\widetilde{Q}$, respectively. Compute the characteristic polynomial of X

$$f(\lambda, \mu) := \det(\lambda M + \mu \widetilde{M}).$$

(2) If $f(\lambda, \mu)$ is irreducible or has an irreducible quartic factor, then $\mathrm{Br}\, X = \mathrm{Br}_0 X$, so give as output $\{\mathrm{id}\}$. Otherwise, for a point $T \in \mathscr{S} := V(f) \subseteq \mathbb{P}^1$, let H_T be a hyperplane that does not contain the vertex of Q_T and let ε_T be the discriminant of $H_T \cap V(Q_T)$.

(3) If there exist three degree 1 points $T_0, T_1, T_2 \in \mathscr{S}$ such that $\varepsilon_{T_i} \notin k^{\times 2}$ and $\varepsilon_{T_i} \varepsilon_{T_j} \in k^{\times 2}$ for all $i, j \in \{0, 1, 2\}$, then choose points $P_i \in V^{\mathrm{smooth}}(Q_{T_i})(k)$. Let ℓ_i be the linear form defining the tangent plane to $V(Q_{T_i})$ at P_i. Give as output

$$\frac{\mathrm{Br}\, X}{\mathrm{Br}_0 X} = \left\{ \mathrm{id}, \left(\varepsilon_{T_0}, \ell_0 \ell_1^{-1}\right), \left(\varepsilon_{T_0}, \ell_0 \ell_2^{-1}\right), \left(\varepsilon_{T_0}, \ell_2 \ell_1^{-1}\right) \right\}.$$

(4) If we get to this step, then $\#(\mathrm{Br}\, X / \mathrm{Br}_0 X) \leq 2$. If there is a k-subscheme $\mathscr{T} \subseteq \mathscr{S}$ that satisfies $(\star)$ and a point $T' \in \mathscr{S} \setminus \mathscr{T}$ such that $\varepsilon_{T'} \notin \kappa(T')^{\times 2}$, then choose points $P_T \in V^{\mathrm{smooth}}(Q_T)(\kappa(T))$ for all $T \in \mathscr{T}$. Let ℓ_T be the linear form defining the tangent plane to $V(Q_T)$ at P_T and let ℓ be any other linear form. Give as output

$$\frac{\mathrm{Br}\, X}{\mathrm{Br}_0 X} = \left\{ \mathrm{id}, \left(\varepsilon_T, \ell^{-2} \prod_{T \in \mathscr{T}} \mathrm{N}_{\kappa(T)/k}(\ell_T)\right) \right\}$$

(5) If we get to this step, then $\operatorname{Br} X = \operatorname{Br}_0 X$, so give as output $\{\mathrm{id}\}$.

Example 4.1. Consider the degree 4 del Pezzo surface X over $\mathbb{Q}$ given by

$$x_0 x_1 + x_2 x_3 + x_4^2 = 0,$$
$$-x_0^2 - 3x_1^2 + x_2^2 - x_2 x_3 + 2x_3^2 + 2x_3 x_4 = 0.$$

The characteristic polynomial is $f(\lambda, \mu) := 2(\lambda^2 - 12\mu^2)(\lambda^3 - 2\lambda^2 \mu - 7\lambda\mu^2 + 4\mu^3)$. Thus $\mathscr{S} = V(f) \subseteq \mathbb{P}^1$ consists of a degree 2 point T and a degree 3 point T'. We may take $H_T = V(x_0)$ and $H_{T'} = V(x_2)$; then $\varepsilon_T = 120\sqrt{3} - 240 = -5 \cdot (2\sqrt{3} - 6)^2$ and $\varepsilon_{T'} = -12\alpha^2 + 40\alpha - 16$, where $\alpha^3 - 2\alpha^2 - 7\alpha + 4 = 0$. Thus, the scheme $\mathscr{T} := \{T\}$ satisfies $(\star)$, and $T' \in \mathscr{S} \setminus \mathscr{T}$ satisfies $\varepsilon_{T'} \notin \kappa(T')^{\times 2}$. Furthermore, $\mathscr{S}(\mathbb{Q}) = \emptyset$, so $\operatorname{Br} X / \operatorname{Br}_0 X = \mathbb{Z}/2\mathbb{Z}$, generated by the class of $\mathcal{A}_{\mathscr{T}}$. Let $P_T = [4 : \sqrt{3} : 1 : 0 : 0] \in V^{\mathrm{smooth}}(Q_T)\left(\mathbb{Q}(\sqrt{3})\right)$; then $\ell_T = 2x_0 - 2x_2 + x_3 - 2\sqrt{3}(x_1 + x_3)$, and finally

$$\mathcal{A}_{\mathscr{T}} = \left(-5, \frac{(2x_0 - 2x_2 + x_3)^2 - 12(x_1 + x_3)^2}{(x_1 + x_3)^2}\right).$$

4.2. **Parameter spaces.** The results in §3 enable us to easily write down parameter spaces whose general points give a del Pezzo surface of degree 4 with a nonconstant Brauer class. Moreover, the largest dimensional such space parametrizes every surface with a nonconstant Brauer class.

For a general degree 4 del Pezzo surface X with a nonconstant Brauer class, the degeneracy locus $\mathscr{S}$ consists of a degree 2 point T and a degree 3 point T'. The following proposition shows that the symmetric matrices associated to X can be given in block form, where the sizes of the blocks correspond to the degrees of the points of $\mathscr{S}$. Thus, we may construct the aforementioned parameter space by first starting with a subvariety of $(\mathbb{P}^{3+6-1})^2$, where each coordinate corresponds to a coefficient of one of the defining quadrics of X.

Proposition 4.2. *Let Q_1 and Q_2 be quadrics in $n+1$ variables over a field k of characteristic not 2, with M_1 and M_2 their respective associated symmetric matrices. Let $\{m_0, m_1, \ldots, m_r\}$ be the degrees of the closed points in $V\left(\det(\lambda M_1 + \mu M_2)\right) \subseteq \mathbb{P}^1$. Assume that $V(Q_1, Q_2) \subseteq \mathbb{P}^n$ is smooth. Then there is a change of variables, defined over k, such that M_1 and M_2 can be written in block diagonal form, where the blocks have sizes $\{m_0, m_1, \ldots, m_r\}$.*

Remark 4.3. The case where $m_j = 1$ for all j already appears in [Wit07, Prop. 3.28]; the argument we give here has the same structure.

Proof. Let T_j denote the closed point in the degeneracy locus of degree m_j and let $v_{i,j} \in \bar{k}^n$ denote the kernels of the corresponding quadrics for $i = 0, \ldots, m_j - 1$. By [Wit07, Prop. 3.28], $\{v_{i,j} : j = 0, \ldots, r, \ i = 0, \ldots, m_j - 1\}$ is an orthogonal basis under Q_1 and Q_2. Let θ_j be a k-algebra generator for $\kappa(T_j)$, let $\tilde{\kappa}(T_j)$ denote the Galois closure of $\kappa(T_j)$, and let $\sigma_j \in \operatorname{Gal}(\tilde{\kappa}(T_j)/k)$ be an order m_j element. Possibly after renumbering, we may assume that $\sigma_j^i(v_{0,j}) = v_{i,j}$. We define

$$w_{i,j} := \sum_{\ell=0}^{m_j-1} \sigma_j^\ell(\theta_j^i) v_{\ell,j}.$$

It is easy to check that $w_{i,j} \in k^n$, that $w_{i,j}$ form a basis, and that if $j \neq j'$, then $w_{i,j}$ and $w_{i',j'}$ are orthogonal under Q_1 and Q_2. Then, the change of basis matrix that takes $\{w_{i,j}\}_{i,j}$ to the standard basis of k^n transforms M_1 and M_2 into block diagonal form. □

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We think of a point in $(\mathbf{a}, \mathbf{b}) \in (\mathbb{P}^{3+6-1})^2$ as two symmetric matrices

$$A := \begin{pmatrix} 2a_{00} & a_{01} & 0 & 0 & 0 \\ a_{01} & 2a_{11} & 0 & 0 & 0 \\ 0 & 0 & 2a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{23} & 2a_{33} & a_{34} \\ 0 & 0 & a_{24} & a_{34} & 2a_{44} \end{pmatrix}, \quad B := \begin{pmatrix} 2b_{00} & b_{01} & 0 & 0 & 0 \\ b_{01} & 2b_{11} & 0 & 0 & 0 \\ 0 & 0 & 2b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{23} & 2b_{33} & b_{34} \\ 0 & 0 & b_{24} & b_{34} & 2b_{44} \end{pmatrix}.$$

After possibly replacing A and B with some linear combinations (which does not affect the corresponding del Pezzo surface), and after possibly making a linear change of coordinates on x_0 and x_1, we may assume that

$$a_{00} = a_{11} = b_{01} = 0.$$

If $a_{01} b_{00} b_{11} = 0$, then the corresponding del Pezzo surface is singular. Hence, we assume that $a_{01} = b_{00} = 1$ and that $b_{11} \neq 0$. By Theorem 3.4, to ensure the existence of a nonconstant Brauer class, it remains to impose that $\mathrm{N}_{\kappa(T)/k}(\varepsilon_T) \in k^{\times 2}$. (The remaining conditions, that ε_T and $\varepsilon_{T'}$ are nonsquares in the fields $\kappa(T)$ and $\kappa(T')$, respectively, are satisfied generically.) To this end, we introduce a new variable w, we let $\widetilde{A}$ and $\widetilde{B}$ be the bottom right 4×4 submatrices of A and B, respectively, and set

$$\mathrm{N}_{\kappa(T)/k}(\varepsilon_T) = \det\left(2\sqrt{-b_{11}}\widetilde{A} + \widetilde{B}\right) \det\left(-2\sqrt{-b_{11}}\widetilde{A} + \widetilde{B}\right) = w^2,$$

which we consider as a double cover of $\mathbb{A}^{13}$. This is the desired parameter space.

Remark 4.4. It is possible to obtain smaller dimensional parameter spaces whose general points correspond to degree 4 del Pezzo surfaces together with one or three distinct nonconstant Brauer classes by further specializing the degree of the points in the degeneracy locus $\mathscr{S}$. These spaces, of course, will not parametrize all del Pezzo surfaces of degree 4 with a nonconstant Brauer class.

4.2.1. *The Brauer group of the generic fiber.* The methods of this paper show that the generic points of the above parameter spaces have nontrivial Brauer class if the associated rank 4 quadrics $V(Q_T)$ have $\kappa(T)$-rational points for all $T \in \mathscr{T}$. There is evidence to suggest that this assumption is necessary absent an assumption on the cohomological dimension of the function field of the parameter space. More precisely, our methods show that for any del Pezzo surface X over any field k of characteristic not 2 such that there exists a $\mathscr{T} \subset \mathscr{S}$ satisfying ($\star$) and a $T \in \mathscr{S} \setminus \mathscr{T}$ with $\varepsilon_T \notin \kappa(T)^{\times 2}$ (e.g., the generic fiber of such parameter spaces), there is always a nontrivial element $\phi \in \mathrm{H}^1(G_K, \mathrm{Pic}\,\mathcal{X}_{\overline{K}})[2]$. However, ϕ arises from a nontrivial class in $\mathrm{Br}_1 X/\mathrm{Br}_0 X$ if and only if it is in the kernel of the boundary map $d_2^{1,1} \colon \mathrm{H}^1(G_k, \mathrm{Pic}\,X_{\overline{k}}) \to \mathrm{H}^3(G_k, \overline{k}^{\times})$ of the Hochschild-Serre spectral sequence.

The map $d_2^{1,1}$ breaks up as the composition of two maps. To see this, we fix some notation. Let $\mathcal{D} \subset \mathrm{Div}\,X_{\overline{k}}$ be a finite Galois invariant set of divisors that generate $\mathrm{Pic}\,X$. Let $R = \ker(\mathcal{D} \to \mathrm{Pic}\,X_{\overline{k}})$, so that the sequences

$$0 \to R \to \mathcal{D} \to \mathrm{Pic}\,X_{\overline{k}} \to 0, \quad \text{and} \quad 0 \to \overline{k}^{\times} \to \mathrm{div}^{-1}(R) \to R \to 0.$$

are short exact. Consider the associated long exact sequences in Galois cohomology. Kresch and Tschinkel [KT08, Proposition 6.1] show that $d_2^{1,1}$ coincides with composition of the

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boundary maps

$$\delta \colon \mathrm{H}^1(G_k, \operatorname{Pic} X_{\overline{k}}) \to \mathrm{H}^2(G_k, R) \quad \text{and} \quad \partial \colon \mathrm{H}^2(G_k, R) \to \mathrm{H}^3(G_k, \overline{k}^{\times}).$$

We now specialize to the case where $\mathscr{S}$ contains two k-rational points T_0 and T_1, $\mathscr{T} := \{T_0, T_1\}$ satisfies $(\star)$, and there exists a $T \in \mathscr{S} \setminus \mathscr{T}$ such that $\varepsilon_T \notin \kappa(T)^{\times 2}$. Then there is a nontrivial element of $\phi \in \mathrm{H}^1(G_k, \operatorname{Pic} X_{\overline{k}})[2]$ corresponding to $\mathscr{T}$. One can show that $\delta(\phi)$ is trivial in $\mathrm{H}^2(G_k, R)$ if and only if both W_{T_0} and W_{T_1} have k-points; this computation does not depend on the field k. Thus, there is hope of finding a field k and a del Pezzo surface X such that $\partial(\delta(\phi))$ is nontrivial in $\mathrm{H}^3(G_k, \overline{k}^{\times})$. In fact, this does occur for cubic surfaces, as shown by recent work of [Uem13]; the above argument borrows ideas from his work.

5. Arithmetic applications

In this section, we restrict to the case where k is a global field of characteristic different from 2. Otherwise, the notation remains as in §2.1. We prove the results stated in the introduction (§§5.1 and 5.2), and explain how the results from §3 can be used to simplify the computation of the $X(\mathbb{A}_k)^{\operatorname{Br}}$ (§5.3). Finally, in §5.4, we give a conditional proof that for certain del Pezzo surfaces of degree 4 over number fields, the set of k-points is dense in the Brauer set.

Lemma 5.1. *Assume that $X(\mathbb{A}_k) \neq \emptyset$. Then $V(Q_T)$ has a smooth $\kappa(T)$-point for all $T \in \mathscr{S}$.*

Proof. Let v be a place of $\kappa(T)$ and let $H \subseteq \mathbb{P}^4$ be a hyperplane that does not contain the vertex of $V(Q_T)$. By assumption, there exists a point $P_v \in X(\kappa(T)_v)$, and thus $P_v \in V(Q_T)$. Since X is smooth, P_v is not the vertex of $V(Q_T)$. Consider the line L that passes through P_v and the vertex of $V(Q_T)$; it is defined over $\kappa(T)_v$, and so L and H intersect in a $\kappa(T)_v$-point P'_v. Clearly, L is contained in $V(Q_T)$ and so $P'_v = L \cap H \subseteq V(Q_T) \cap H$. Thus $(V(Q_T) \cap H)(\mathbb{A}_{\kappa(T)}) \neq \emptyset$. Since smooth quadrics in at least 3 variables satisfy the Hasse principle, this completes the proof. $\square$

Therefore, if $X(\mathbb{A}_k) \neq \emptyset$, then we may apply the results of §3.

5.1. **Proof of Theorem 1.1.** Most of the theorem follows immediately from Theorem 3.4 and Corollary 3.5. It remains to show that if $\#(\operatorname{Br} X / \operatorname{Br}_0 X) = 4$, then there is a map $f \colon X \dashrightarrow \mathbb{P}^2$ such that $\operatorname{Br} X = \operatorname{Br}_{\operatorname{vert}}^{(f)} X$. By Theorem 3.4, there exist three k-points T_0, T_1 and $T_2 \in \mathscr{S}$ such that every pair satisfies $(\star)$. Let ℓ_i be a k-linear form such that the associated hyperplane is tangent to $V(Q_{T_i})$ at a smooth point. Then by Theorem 3.4 and the definition of $\mathcal{A}_{\mathscr{T}}$, the map

$$f \colon X \dashrightarrow \mathbb{P}^2, \qquad x \mapsto [\ell_0(x) : \ell_1(x) : \ell_2(x)]$$

has the desired property. $\square$

5.2. **Proof of Corollary 1.3.** We assume a certain familiarity with the Brauer-Manin obstruction (see [Sko01, §5.2] for a detailed treatment of this topic). Recall that for any $\mathcal{S} \subseteq \operatorname{Br} X$, the Brauer-Manin pairing gives rise to a subset $X(\mathbb{A}_k)^{\mathcal{S}} \subseteq X(\mathbb{A}_k)$. Class field theory shows that the set $X(\mathbb{A}_k)^{\mathcal{S}}$ is closed in $X(\mathbb{A}_k)$ for the adelic topology, and it contains $X(k)$. In keeping with standard notation, when $\mathcal{S} = \{\mathcal{A}\}$, $\operatorname{Br}_{\operatorname{vert}}^{(g)} X$ or $\operatorname{Br} X$, respectively, we write $X(\mathbb{A})^{\mathcal{A}}$, $X(\mathbb{A}_k)^{\operatorname{Br}_{\operatorname{vert}}^{(g)}}$, or $X(\mathbb{A}_k)^{\operatorname{Br}}$ for $X(\mathbb{A}_k)^{\mathcal{S}}$.

In preparation for the proof of Corollary 1.3, we state two results.

Proposition 5.2. *Let k be a number field and consider a projection from a plane $g\colon X \to \mathbb{P}^1$ with at most 2 geometrically reducible fibers. Then*

$$\overline{\{t \in \mathbb{P}^1(k) : g^{-1}(t)(\mathbb{A}_k) \neq \emptyset \text{ and } g^{-1}(t) \text{ smooth}\}} = g(X(\mathbb{A}_k)^{\mathrm{Br}^{(g)}_{\mathrm{vert}}})$$

where the closure takes place in the adelic topology.

Proof. If the geometrically reducible fibers lie over k-points then this follows from [CTSSD98b, Thm. 2.2.1(a)]. Otherwise it follows from [Sko96, proof of Thm. 0.4]. $\qquad\square$

Proposition 5.3 ([CTP00, Lemma 3.4 and Remark 2 p. 95])**.** *Let k be a number field. Then $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ if and only if there exists an $\mathcal{A} \in \mathrm{Br}\, X$ such that $X(\mathbb{A}_k)^{\mathcal{A}} = \emptyset$.* $\qquad\square$

Proof of Corollary 1.3. The second claim follows immediately from Theorem 1.1 and Proposition 5.2 so we focus on the first claim.

By Proposition 5.2, if there exists a map $g\colon X \dashrightarrow \mathbb{P}^1$, obtained by projecting from a plane, with at most two geometrically reducible fibers such that

$$g^{-1}(t)(\mathbb{A}_k) = \emptyset \quad \forall t \in \mathbb{P}^1(k),$$

then $X(\mathbb{A}_k)^{\mathrm{Br}^{(g)}_{\mathrm{vert}}} = \emptyset$, which in turn implies that $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. For the forward implication, we note that Proposition 5.3 and Theorem 1.1 show that

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset \implies X(\mathbb{A}_k)^{\mathrm{Br}^{(g_{\mathcal{A}})}_{\mathrm{vert}}} = \emptyset \text{ for some } \mathcal{A} \in \mathrm{Br}\, X.$$

If $X(\mathbb{A}_k) = \emptyset$, then it is immediate that there exists a map $g\colon X \dashrightarrow \mathbb{P}^1$ such that $g^{-1}(t)(\mathbb{A}_k) = \emptyset$ for all $t \in \mathbb{P}^1(k)$. Thus we may assume that $X(\mathbb{A}_k) \neq \emptyset$ and that $\mathcal{A} \notin \mathrm{Br}\, k$.

Let $g = g_{\mathcal{A}}\colon X \dashrightarrow \mathbb{P}^1$ be as in Corollary 3.5. Assume that there exists $t \in \mathbb{P}^1(k)$ such that $g^{-1}(t)(\mathbb{A}_k) \neq \emptyset$; we show this implies that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Fix a point $(P_v) \in g^{-1}(t)(\mathbb{A}_k)$, and let $U \subseteq \mathbb{P}^1$ be the largest Zariski open set such that $\mathrm{Br}\, X \subseteq g^*(\mathrm{Br}\, U)$. Then for any $\mathcal{A} \in \mathrm{Br}\, X$ there exists a $\mathcal{B} \in \mathrm{Br}\, U$ such that $g^*\mathcal{B} = \mathcal{A}$.

Suppose that $t \in U$. Let $\mathrm{inv}_v\colon \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z}$ be the local invariant map from local class field theory. Functoriality of the Brauer group and local class field theory imply that

$$\sum_v \mathrm{inv}_v(\mathcal{A}(P_v)) = \sum_v \mathrm{inv}_v((g^*\mathcal{B})(P_v)) = \sum_v \mathrm{inv}_v(\mathcal{B}(g(P_v))) = \sum_v \mathrm{inv}_v(\mathcal{B}(t)) = 0$$

and thus $(P_v) \in X(\mathbb{A}_k)^{\mathrm{Br}}$.

Now suppose that $t \notin U$, i.e., that t is in the ramification locus of some $\mathcal{B} \in \mathrm{Br}\, \mathbf{k}(\mathbb{P}^1) \setminus \mathrm{Br}\, k$ such that $g^*\mathcal{B} \in \mathrm{Br}\, X$. By Theorem 3.4, we may assume that $\mathcal{B} = \mathcal{A}_{\mathscr{T}}$ for some $\mathscr{T} \subseteq \mathscr{S}$ satisfying $(\star)$. Since $t \in \mathbb{P}^1(k)$, by construction of $\mathcal{A}_{\mathscr{T}}$, we must have that $\#\mathscr{T}(k) = 2$. Additionally, the inverse image of an irreducible component of the ramification locus of $\mathcal{A}_{\mathscr{T}}$ (considered as an element of $\mathrm{Br}\, \mathbf{k}(\mathbb{P}^1)$) is the union of two irreducible curves, each defined over a nontrivial quadratic extension of k, that meet in exactly two points. By assumption, since $g^{-1}(t)(\mathbb{A}_k) \neq \emptyset$, the union of these two curves contains adelic points, and thus the curves meet in two k-points. Hence $X(\mathbb{A}_k)^{\mathrm{Br}} \supseteq X(k) \neq \emptyset$. $\qquad\square$

5.3. Computation of the evaluation map. If X is a degree 4 del Pezzo surface over a number field, then for any $\mathcal{A} \in \mathrm{Br}\, X$ and any place v of good reduction, the evaluation map $ev_{\mathcal{A}}\colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is constant [CTS12, Bri07]. The results of §3 allow us to conclude the same is true for a larger set of places, over global fields of characteristic not 2.

Proposition 5.4. *Let $v \nmid 2$ be a place of k, and let $\mathscr{T} \subseteq \mathscr{S}$ satisfy $(\star)$. If, for all $T \in \mathscr{T}$, Q_T modulo v is a rank 4 quadric, then for any $\mathcal{B} \in \operatorname{Br} X$ such that $\mathcal{B} = \mathcal{A}_{\mathscr{T}}$ in $\operatorname{Br} X / \operatorname{Br}_0 X$, the evaluation map $ev_{\mathcal{B}} \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is constant.*

Proof. Write $\mathbb{F}_v$ for the residue field of k_v. Since Q_T modulo v is a rank 4 quadric, by Lemma 2.1, there exist k-linear forms $f_{T,1}, f_{T,2}, f_{T,3}, f_{T,4}$ that are linearly independent over $\mathbb{F}_v$ and $c \in k^\times$ such that
$$cQ_T = f_{T,1}f_{T,2} - (f_{T,3}^2 - \varepsilon_{T_0} f_{T,4}^2).$$
Therefore, for $i_T \in \{1, 2\}$, the algebras $\left(\varepsilon_{T_0}, \ell^{-2} \prod_{T \in \mathscr{T}} \mathrm{N}_{\kappa(T)/k}(f_{T,i_T}) \right)$ are isomorphic; we write $\mathcal{A}$ for this isomorphism class. Note that $\mathcal{A}$ is equal to $\mathcal{A}_{\mathscr{T}}$ in $\operatorname{Br} X / \operatorname{Br}_0 X$.

If $\varepsilon_{T_0} \in k_v^{\times 2}$, then $\mathcal{A}$ is trivial when considered as an element of $\operatorname{Br} X_v$, and so the evaluation map $ev_{\mathcal{A}}$ is identically zero. Assume that $\varepsilon_{T_0} \notin k_v^{\times 2}$; since $v \nmid 2$ and Q_T does not drop rank modulo v, the extension $k(\sqrt{\varepsilon_{T_0}})/k$ is unramified at v. By properness of X, we know that $X(k_v) = X(\mathcal{O}_v)$. Then for any $P \in X(\mathcal{O}_v) \subseteq V(Q_T)^{\mathrm{smooth}}(\mathcal{O}_v)$, at least one of the $f_{T,i}(P)$ is in $\mathcal{O}_v^\times$. If $f_{T,3}(P)$ or $f_{T,4}(P)$ is a v-adic unit, then, since $\varepsilon_{T_0} \notin k_v^{\times 2}$, both $f_{T,1}(P)$ and $f_{T,2}(P)$ are v-adic units. A Hilbert symbol calculation then shows that $ev_{\mathcal{A}}(P) = 0$.

For any $\mathcal{B} \in \operatorname{Br} X$ such that $\mathcal{B} = \mathcal{A}_{\mathscr{T}}$ in $\operatorname{Br} X / \operatorname{Br}_0 X$, we have $\mathcal{B} - \mathcal{A} \in \operatorname{Br}_0 X$; since $ev_{\mathcal{A}}$ is identically zero, it follows that $ev_{\mathcal{B}}$ is constant. $\qquad\square$

5.4. Density of k-points in the Brauer set. In this section, we prove that certain del Pezzo surfaces of degree 4 over a number field k satisfy $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$, i.e., the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one on X. This result is conditional on Schinzel's hypothesis (see [CTSD94, §4] for its statement over number fields) and the finiteness of Tate-Shafarevich groups of elliptic curves. Some of the cases we consider were already known by [Wit07, Théorème 3.36(iv)], [CTSSD98, Proposition 3.2.1(c)], and [HSW, Theorem 5.1]; the latter result notably does not require the use of Schinzel's hypothesis. However, our results also include some new cases, e.g., some degree 4 del Pezzo surfaces whose associated pencil of quadrics contains 3 k-rational rank 4 quadrics that are not necessarily simultaneously diagonalizable.

We consider del Pezzo surfaces X of degree 4 over a number field k with the following properties:

(1) the scheme $\mathscr{S}$ contains three k-rational points T_0, T_1, T_2;
(2) the subscheme $\mathscr{T} := \{T_0, T_1\}$ of $\mathscr{S}$ satisfies $(\star)$ while any other degree 2 subschemes of $\mathscr{S}$ containing either T_0 or T_1 do not; and
(3) the line spanned by the vertices of the rank 4 quadrics W_3 and W_4 intersects W_0 and W_1 each in two k-rational points.

Since the surface (1.1), considered by Birch and Swinnerton-Dyer in [BSD75], satisfies these conditions, we say that such a surface is of **BSD type**. In fact, we will see that any surface of BSD type has a form similar to that of (1.1), further justifying the nomenclature.

By Proposition 4.2, we may assume that a del Pezzo surface of degree 4 satisfying (1) is given by the intersection of two quadrics of the following form:
$$Q_0'(x_3, x_4) = x_2^2 - \varepsilon_0 d_0 x_1^2,$$
$$Q_1'(x_3, x_4) = x_2^2 - \varepsilon_1 d_1 x_0^2,$$
where d_i has the same square class as the discriminant of Q_i'. Condition (2) allows us to assume that $\varepsilon_0 = \varepsilon_1$, and further implies, by Theorem 3.4, that $\operatorname{Br} X / \operatorname{Br} k$ is generated

by $\mathcal{A}_{\mathscr{T}}$. Lastly, condition (3) allows us to assume that Q_0' and Q_1' are split, and thus that $d_0 = d_1 = 1$. After a change of coordinates, we may assume that $Q_0' = cx_3x_4$ and that $Q_1' = (x_3 + x_4)(ax_3 + bx_4)$. In summary, any degree 4 del Pezzo surface of BSD type is the intersection of two quadrics of the following form:

$$cx_3x_4 = x_2^2 - \varepsilon x_1^2,$$
$$(x_3 + x_4)(ax_3 + bx_4) = x_2^2 - \varepsilon x_0^2. \tag{5.1}$$

Theorem 5.5 (Also Theorem 1.5). *Let X be a del Pezzo surface of degree 4 over a number field k of BSD-type. Assume Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups of elliptic curves. Then $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction to the Hasse principle and weak approximation on X is the only one.*

Proof. If $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, then there is nothing to prove. We will show that if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ then $X(k) \neq \emptyset$. Work of Salberger and Skorobogatov [SS91, Theorem 0.1] then implies that $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$. In [Wit07, Théorème 1.1], Wittenberg shows that the Brauer-Manin obstruction to the Hasse principle is the only one for genus 1 fibrations $Y \to \mathbb{P}^1$ such that

(i) the generic fiber Y_η has period 2, i.e., it has no $\mathbf{k}(\mathbb{P}^1)$-rational points, but acquires a rational point over a quadratic extension of $\mathbf{k}(\mathbb{P}^1)$,
(ii) the Jacobian $\mathrm{Jac}\, Y_\eta$ has $\mathbf{k}(\mathbb{P}^1)$-rational 2-torsion, and
(iii) the so-called "condition (D)" is satisfied (we explain this condition below).

Therefore, to prove the theorem, it suffices to construct a genus 1 fibration on (a birational model of) X, and prove that it has the desired properties. Throughout, we assume that X is of the form (5.1).

Let $Y := \mathrm{Bl}_{V(x_3,x_4)}X$, and write $E_1, E_2, E_3,$ and E_4, respectively, for the exceptional curves of the blow-up lying over the points

$$[1 : 1 : \sqrt{\varepsilon} : 0 : 0], \ [-1 : 1 : \sqrt{\varepsilon} : 0 : 0], \ [1 : -1 : \sqrt{\varepsilon} : 0 : 0], \ \text{and} \ [-1 : -1 : \sqrt{\varepsilon} : 0 : 0]$$

of $V(x_3, x_4) \cap X$. Consider the fibration $f' \colon Y \to \mathbb{P}^1$ that sends $\mathbf{x} \mapsto [x_3 : x_4]$. Since the generic fiber of f' is the intersection of two quadrics in $\mathbb{P}^3$, this is a genus 1 fibration. We claim that f' has the desired properties.

By Proposition 2.2 and standard properties of blow-ups, $\mathrm{Pic}\,\overline{Y}$ is freely generated by

$$C_0, \ C_1, \ C_2, \ C_3, \ C_4, \ B := \frac{1}{2}\left(H + C_0 + C_1 + C_2 + C_3 + C_4\right), \ E_1, \ E_2, \ E_3, \ \text{and} \ E_4,$$

To determine the Picard group of $\overline{Y}_\eta$, we must determine the reducible fibers of f' and their classes in $\mathrm{Pic}\,\overline{Y}$.

By [CTSSD87, §1], the intersection of X with a hyperplane H is reducible if and only if H is tangent to a rank 4 quadric at a smooth point. Using this fact, we see that the reducible fibers of f' lie above

$$V(st(s + t)(as + bt)(as^2 + (a + b - c)st + bt^2)) \subset \mathbb{P}^1_{(s:t)}.$$

Using the discussion in §2.3 and standard properties of the blow-up, the class of a general fiber of f' is $F := H - E_1 - E_2 - E_3 - E_4$, and each reducible fiber contains a component whose class is among the following

$$C_0 - E_1 - E_2, \ C_0 - E_3 - E_4, \ C_1 - E_1 - E_3, \ C_1 - E_2 - E_4, \ C_2 - E_1 - E_4, \ C_2 - E_2 - E_3.$$

Together, these 7 classes generate the kernel of the restriction map $\mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{Y}_\eta$. Thus, for all $i \neq j$, $2E_i - 2E_j$ is trivial in $\mathrm{Pic}\,\overline{Y}_\eta$ and $E_i - E_j$ is non-trivial in $\mathrm{Pic}\,\overline{Y}_\eta$. Additionally, $E_1 + E_2 - E_3 - E_4$ is trivial in $\mathrm{Pic}\,\overline{Y}_\eta$, and therefore $\left(\mathrm{Jac}\,\overline{Y}_\eta\right)[2] = \langle E_1 - E_2, E_1 - E_3 \rangle$. Since each element of the absolute Galois group G_k either fixes all exceptional curves, or simultatenously interchanges E_1 with E_4 and E_2 with E_3, we deduce that every 2-torsion class in $\mathrm{Jac}\,\overline{Y}_\eta$ is $\mathbf{k}(\mathbb{P}^1)$-rational. In other words, f' satisfies condition (ii).

Next, we turn to condition (i). The class $E_1 + E_4$ in $\mathrm{Pic}\,\overline{Y}$ is Galois invariant and gives rise to a degree 2 point of the generic fiber Y_η; therefore the period of Y_η divides 2. It remains to show that the period of Y_η is not 1. Consider the following intersection numbers:

$$F \cdot E_i = 1, \quad F \cdot C_j = 2, \quad \text{and } F \cdot B = 7.$$

If Y_η has a $\mathbf{k}(\mathbb{P}^1)$-rational point, then there is a Galois invariant degree 1 element of $\mathrm{Pic}\,\overline{Y}_\eta$. Since the restriction map $\mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{Y}_\eta$ is surjective, we may compute $\mathrm{Pic}\,\overline{Y}_\eta$, as a Galois module, using the generators and relations given above. In particular, we deduce that $\mathrm{Pic}\,\overline{Y}_\eta$ is generated by

$$B, \ C_3, \ E_3, \ (E_1 - E_3), \ \text{and } (E_2 - E_3),$$

with relations $2(E_2 - E_3) = 2(E_1 - E_3) = 0$. By BSD type condition (2), there exists an element $\sigma \in G_k$ such that $\sigma(C_0) = C_0'$, $\sigma(C_1) = C_1'$ and $\sigma(C_2) = C_2$; this also implies that $\sigma(E_3) = E_2$. One can check that, regardless of the action of σ on C_3 and C_4, we have

$$\sigma(B) = \beta B + \gamma C_3 + \delta E_3 + (E_2 - E_3), \ \sigma(C_3) = \beta' B + \gamma' C_3 + \delta' E_3, \ \text{and } \ \sigma(E_3) = E_3 + (E_2 - E_3)$$

for some integers $\beta, \beta', \gamma, \gamma', \delta$ and δ'. Therefore, if some integer combination of $B, C_3, E_3, (E_2 - E_3)$ and $(E_1 - E_3)$ is fixed by G_k, then the integer coefficients of B and E_3 must have the same parity. However, such a divisor class has even intersection with F, and so there is no Galois invariant divisor of degree 1. In particular, this implies that Y_η has period equal to 2 and condition (i) is satisfied.

Finally, we show that f' satisfies condition (D). Let $[\alpha : 1] \in \mathbb{P}^1_{(s:t)}$ be such that $f'^{-1}([\alpha : 1])$ is smooth and set $U := \mathbb{P}^1 \setminus [\alpha : 1]$. Let $\mathrm{Br}^{\mathrm{gnr}/Y}\,Y_U$ denote the group of classes in $\mathrm{Br}\,Y_U$ generated by classes *geometrically* unramified over Y. By [Wit07, Corollaire 1.48], condition (D) on the fibration f' is implied by the inclusion

$$\left(\mathrm{Br}^{\mathrm{gnr}/Y}\,Y_U\right)\{2\} \subset \mathrm{Br}^{(f')}_{\mathrm{vert}}\,Y_U. \tag{5.2}$$

A few words are in order, since we do not check one of the hypotheses in [Wit07, Corollaire 1.48]. In Wittenberg's notation, the hypothesis that $[\mathscr{X}]$ is not divisible by 2 is not necessary to deduce that (5.2) implies condition (D); it is only used to prove the converse statement. Indeed, if (5.2) holds then by [Wit07, Corollaire 1.47] we have $\mathscr{D}(\mathbb{P}^1_k, Y)[2] = \left(\mathrm{Br}^{\mathrm{gnr}/Y}_{\mathrm{hor}}\,Y_U\right)[2] = 0$. By [Wit07, Corollaire 1.43], it follows that $\mathfrak{T}_{D/\mathbb{P}^1_k}/\langle [\mathscr{X}] \rangle = 0$, which is to say that condition (D) holds; see [Wit07, pp. 24–25]. The hypothesis that $L_M = \kappa(M)$ for all $M \in \mathscr{M}$ in [Wit07, Corollaire 1.48] is also satisfied; see [Wit07, Corollaire A.5].

Since Y is geometrically rational, we have $\mathrm{Br}^{\mathrm{gnr}/Y}\,Y_U = \mathrm{Br}_1\,Y_U$. It is thus enough to show that

$$\frac{\mathrm{Br}_1\,Y_U}{\mathrm{Br}\,k}\{2\} \subset \frac{\mathrm{Br}^{(f')}_{\mathrm{vert}}\,Y_U}{\mathrm{Br}\,k},$$

By the sequence of low-degree terms of the Hochschild-Serre spectral sequence, the group $\mathrm{Br}_1 Y_U / \mathrm{Br}\, k$ is isomorphic to $\mathrm{H}^1\left(G_k, \mathrm{Pic}\, Y_{\overline{U}}\right)$. Note that $\mathrm{Pic}\, Y_{\overline{U}} \cong \mathrm{Pic}\, Y_{\overline{k}}/\langle F\rangle$ as Galois modules, so we can use our explicit generators for $\mathrm{Pic}\,\overline{Y}$ to compute this cohomology group, once we know the Galois actions on these generators. Since X is of BSD type, the sets

$$\{E_1, E_4\},\ \{E_2, E_3\},\ \{C_0, C_0'\},\ \{C_1, C_1'\},\ \{C_2, C_2'\},\ \text{ and }\ \{C_3, C_3', C_4, C_4'\}$$

are all invariant under G_k, and by (3.3), the definition of E_i and ($\star$), the action of G_k on $\{C_0, C_0'\}$, and $\{C_3, C_3', C_4, C_4'\}$ determines the action on $\{C_1, C_1'\}, \{C_2, C_2'\}, \{E_1, E_4\}$, and $\{E_2, E_3\}$. Therefore, the action of G_k factors through a subgroup of $\mathbb{Z}/2 \times D_4$, and, in each case, one can compute, either using Magma [BCP97] or by hand using the last isomorphism in (3.1), that

$$\mathrm{H}^1\left(G_k, \mathrm{Pic}\, Y_{\overline{U}}\right) \cong \mathrm{H}^1\left(G_k, \mathrm{Pic}\, X_{\overline{k}}\right) \times \mathbb{Z}/2.$$

On the other hand, $(\varepsilon, 1 - \alpha x_4/x_3) \in \mathrm{Br}\, Y_U$ and, since it is ramified on F, it is nontrivial in $\mathrm{Br}\, \mathbf{k}(Y)$ and not contained in $\mathrm{Br}\, Y$. Therefore, $\frac{\mathrm{Br}_1 Y_U}{\mathrm{Br}\, k}$ is generated by $\mathcal{A}_{\mathscr{T}} = (\varepsilon, x_4/x_3 + 1)$ and $(\varepsilon, 1 - \alpha x_4/x_3)$, both of which are plainly vertical. Hence condition (D) holds for f', as desired, which completes the proof. $\square$

6. Brauer groups of order 4

It is natural to ask whether Theorem 1.1 can be strengthened to show that $\mathrm{Br}\, X = \mathrm{Br}_{\mathrm{vert}}^{(f)} X$ for some map $f\colon X \dashrightarrow \mathbb{P}^1$, even if $\#(\mathrm{Br}\, X/\mathrm{Br}_0 X) = 4$. In this section we consider a variation of this question for maps obtained by projecting away from a plane in $\mathbb{P}^4$.

We retain the notation from §2; throughout, we assume that $\#(\mathrm{Br}\, X/\mathrm{Br}_0 X) = 4$. By Theorem 3.4, there exist three k-points T_0, T_1, and $T_2 \in \mathscr{S}(k)$ such that each pair satisfies ($\star$). In this case, the three nontrivial elements of $\mathrm{Br}\, X/\mathrm{Br}_0 X$ are represented by the quaternion algebras

$$\{(\varepsilon_{T_0}, \ell_i/\ell_j) : 0 \le i < j \le 2\},$$

where ℓ_i is any k-linear form whose corresponding hyperplane is tangent to $V(Q_{T_i})$ at a smooth point.

We ask if there are forms ℓ_0, ℓ_1 and ℓ_2 as above and a rational map $f\colon X \dashrightarrow \mathbb{P}^1$, obtained by projecting away from a plane, such that $\ell_i/\ell_j \in f^*(\mathbf{k}(\mathbb{P}^1))$. If so, then $\mathrm{Br}\, X = \mathrm{Br}_{\mathrm{vert}}^{(f)} X$ for this map. If $\ell_i/\ell_j \in f^*(\mathbf{k}(\mathbb{P}^1))$, then, possibly after changing coordinates, the map f is given by $x \mapsto [\ell_i(x) : \ell_j(x)]$. Thus, a positive answer to the question is equivalent to k-linear dependence of ℓ_0, ℓ_1, and ℓ_2.

Proposition 6.1. *Let X be a del Pezzo surface of degree 4 over k. Assume that $X(k) \ne \emptyset$ and that $\#(\mathrm{Br}\, X/\mathrm{Br}_0 X) = 4$. Then there exists a map $f\colon X \dashrightarrow \mathbb{P}^1$, obtained by projection away from a plane, such that $\mathrm{Br}\, X = \mathrm{Br}_{\mathrm{vert}}^{(f)} X$.*

Proof. Let $P \in X(k)$ and let ℓ_0, ℓ_1 be linear forms such that the corresponding hyperplanes are tangent to $V(Q_{T_0})$ and $V(Q_{T_1})$ at P, respectively. Define the map

$$f\colon X \dashrightarrow \mathbb{P}^1 \qquad x \mapsto [\ell_0(x) : \ell_1(x)].$$

Since Q_{T_2} is a k-linear combination of Q_{T_0} and Q_{T_1}, there is a k-linear combination ℓ_2 of ℓ_0 and ℓ_1 whose corresponding hyperplane is tangent to $V(Q_{T_2})$ at P. The definition of $\mathcal{A}_{\mathscr{T}}$ and Theorem 3.4 show that $\mathrm{Br}\, X = \mathrm{Br}_{\mathrm{vert}}^{(f)} X$. $\square$

Remark 6.2. The proof of Proposition 6.1 gives a map $f\colon X \dashrightarrow \mathbb{P}^1$ whose generic fiber is singular. To see this, let ℓ be a k-linear combination of ℓ_0 and ℓ_1. Then there is some quadric Q in the pencil of quadrics defining X such that $V(\ell)$ is tangent to $V(Q)$ at a smooth point P. The Jacobian criterion then shows that $X \cap V(\ell)$ is singular at P.

One wonders if Proposition 6.1 is as strong as possible: does the conclusion hold without assuming that $X(k) \neq \emptyset$? Is it possible to construct a map f whose generic fiber is not singular? After rephrasing these questions more precisely, we will see that they are essentially the same question.

The linear dependence condition on ℓ_0, ℓ_1, and ℓ_2 gives rise to a determinantal subvariety $Y^{(X)}$ in a product of three quadric surfaces, as follows. Say $Q_{T_i} \in k[x_0, \ldots, x_4]$, and let $P_i \in \mathbb{P}^4(\overline{k})$ for $i = 0, 1, 2$. Consider the matrix $M(P_0, P_1, P_2)$ whose (i, j)-th entry is

$$\frac{\partial Q_{T_i}}{\partial x_j}(P_i).$$

Let $H \subseteq \mathbb{P}^4$ be a hyperplane avoiding the vertices of $V(Q_{T_i})$ for $0 \leq i \leq 2$. Then define

$$Y^{(X)} := \{(P_0, P_1, P_2) \in H^3 : \operatorname{rk} M(P_0, P_1, P_2) \leq 2 \text{ and } Q_{T_i}(P_i) = 0\}.$$

The existence of k-linearly dependent forms ℓ_0, ℓ_1, and ℓ_2 is equivalent to $Y^{(X)}(k) \neq \emptyset$: if $(P_0, P_1, P_2) \in Y^{(X)}(k)$, then we take ℓ_i to be the linear form defining the hyperplane tangent to $V(Q_{T_i})$ at P_i, for $0 \leq i \leq 2$.

There is an embedding of $X(k)$ in $Y^{(X)}(k)$: for a point $x \in X(k)$, let $P_i(x) \in H(k)$ be the intersection of H with the line joining x and the vertex of $V(Q_{T_i})$. Then $(P_0(x), P_1(x), P_2(x))$ is a k-point of $Y^{(X)}$; this is the point giving rise to the map f described in the proof of Proposition 6.1. From this discussion we see that the questions above are equivalent to the following:

Question 6.3. *Is there a k-rational point on $Y^{(X)}$ that does not come from the embedding of $X(k)$ as described above?*

Remark 6.4. If we relax the condition that f is obtained by projecting away from a plane, then the existence of a k-point implies that $\operatorname{Br} X = \operatorname{Br}_{\mathrm{vert}}^{(f)} X$ where f is a generically smooth map to $\mathbb{P}^1$. The idea is as follows.

Let $P \in X(k)$ and consider $\widetilde{X}$, the blow-up of X at P. The surface $\widetilde{X}$ is a del Pezzo surface of degree 3 containing a line L defined over k. Embed $\widetilde{X}$ into $\mathbb{P}^3$ by the anticanonical embedding and consider the pencil of planes that contain L. This gives a rational map $f\colon X \dashrightarrow \mathbb{P}^1$ whose generic fiber is a smooth conic. Since the generic fiber is a smooth conic, we have $\operatorname{Br} X = \operatorname{Br}_{\mathrm{vert}}^{(f)}(X)$.

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