ALBANESE MAPS AND FUNDAMENTAL GROUPS OF VARIETIES WITH MANY RATIONAL POINTS OVER FUNCTION FIELDS

These are notes for the talk *Albanese maps and fundamental groups of varieties with many rational points over function fields* given at WAGON on Sunday April 19th. The work we present is joint work (in progress) with Erwan Rousseau (Marseille).

1. Campana's conjecture on special varieties

Throughout these notes, let X be a smooth projective connected variety over $\mathbb{C}$.

A surjective morphism $X \to Y$ with connected fibers is called a fibration (on X). We say that a fibration $X \to Y$ is of *general type* if $\omega_Y(\Delta_f)$ is big and Y has dimension > 0, where Δ_f is a certain effective $\mathbb{Q}$-divisor encoding the multiple fibres of $X \to Y$. In particular, if Y is of general type, then a fibration $X \to Y$ is of general type.

Definition 1.1 (Campana)**.** We say that X is *special* if every fibration on X is **not** of general type.

Campana originally defined a variety to be special if it has no Bogomolov sheaves. The definition we give here is a bit quicker to state, and also immediately clarifies (geometrically) that being "special" is opposite to being "of general type". One also sees immediately that a special variety does not dominate a variety of general type:

Remark 1.2. A special variety does not dominate a positive-dimensional variety of general type. (Thus, for C a curve of genus at least two, $\mathbb{P}^1 \times C$ is neither special nor of general type.)

Proposition 1.3. *If* $\dim X = 1$*, then* X *is special if and only if* $\mathrm{genus}(X) \leqslant 1$*. That is,* X *is special if and only if* X *is a rational curve or an elliptic curve.*

Example 1.4. If X is rationally connected, then X is special.

Example 1.5. If X is an abelian variety, then X is special. (Actually, if X has Kodaira dimension zero, then X is special.)

Campana conjectured that there is an analytic characterisation of special varieties. His conjecture reads as follows.

Conjecture 1.6 (Campana)**.** *The variety* X *is special if and only if it has a dense entire curve, i.e., there is a non-constant holomorphic map* $\mathbb{C} \to X^{an}$ *whose image is Zariski-dense.*

Campana's conjecture is "opposite" to the Green-Griffiths-Lang conjecture which says that X is of general type if and only if there is a proper closed subset $D \subset X$ such that every non-constant holomorphic map $\mathbb{C} \to X^{\mathrm{an}}$ factors over D^{an}. (By viewing "special" as opposite to "general type" one is naturally led to Campana's above conjecture.)

Campana's conjecture holds for curves. This is essentially a consequence of the Riemann uniformization theorem that curves of genus at least two are uniformised by the open unit disc in $\mathbb{C}$ (and thus have no entire curves).

Theorem 1.7. *If* $\dim X = 1$*, then* X *is special if and only if genus*$(X) = 0, 1$ *if and only if* X *has a dense entire curve.*

Example 1.8. If X is an abelian variety, then X has a dense entire curve. (Use that $\mathbb{C}^{\dim X}$ uniformises X.)

More recently, Campana-Winkelmann verified that rationally connected varieties (which are special) verify Campana's conjecture. Indeed:

Theorem 1.9 (Campana-Winkelmann, 2019)**.** *If* X *is rationally connected, then* X *has a dense entire curve.*

Remark 1.10. It is not known whether every K3 surface has a dense entire curve (unless the K3 surface admits an elliptic fibration, for example). However, there is some evidence that such a dense entire curve exists. For example, Verbitsky showed that the Kobayashi pseudometric on a K3 surface is identically zero.

The properties of special varieties are not fully understood yet. For example, concerning the topological fundamental group $\pi_1(X) := \pi_1(X^{\mathrm{an}})$ of a special variety X, we have the following conjecture of Campana. (A group is called *virtually abelian* if it contains a finite index abelian subgroup.)

Conjecture 1.11 (Campana)**.** *If* X *is special, then* $\pi_1(X)$ *is virtually abelian.*

If X is an abelian variety or X is rationally connected, then this conjecture is true. Indeed, if X is an abelian variety, then $\pi_1(X)$ is a (finitely generated free) abelian group and if X is rationally connected then $\pi_1(X)$ is trivial (Campana, Kollár-Miyaoka-Mori).

The main result of the project presented today is motivated by Campana's conjecture and the following two results of Campana and Yamanoi, respectively.

Theorem 1.12 (Campana)**.** *If* X *is special, then every linear quotient of* $\pi_1(X)$ *is virtually abelian, i.e,. for every* $\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$*, the image of* ρ *is virtually abelian.*

Thus, if X is special and the fundamental group of X is a linear group, then $\pi_1(X)$ is virtually abelian.

Motivated by Campana's analytic conjectural characterization of special varieties, Yamanoi proved a similar theorem for varieties with a dense entire curve.

Theorem 1.13 (Yamanoi)**.** *If X has a dense entire curve, then every linear quotient of $\pi_1(X)$ is virtually abelian.*

We now introduce a notion of "specialness" for constant varieties over a function field.

2. Function field analogue

This is joint work with Erwan Rousseau.

Recall that X denotes a smooth projective connected variety over $\mathbb{C}$.
Campana conjectured that, for $\mathcal{X}$ a variety over a finitely generated field k with $\mathcal{X}_{\overline{k}}$ is special, the set $\mathcal{X}(L)$ should be dense for some finite field extension L/k. (This is true for curves of genus zero and one. This property also holds for abelian varieties. However, it is not known for rationally connected varieties, nor for all K3 surfaces.) Note that this arithmetic conjecture of Campana is "opposite" to Lang's conjecture on rational points which says that X is of general type if and only if there is a proper closed subset $\Delta \subsetneq X$ such that, for every finitely generated subfield $k \subset \mathbb{C}$ and every model $\mathcal{X}$ for X over k, the set $\mathcal{X}(k) \setminus \Delta$ is finite.

Motivated by Campana's conjectures on special varieties, Lang's conjecture on hyperbolicity and the analogy between number fields and function fields, we introduce the following "function field" analogue:

Definition 2.1 (J.-Rousseau)**.** We say that X is *geometrically-special* if, for every dense open U, there is a point x in U, a pointed curve (C, c), and a sequence of maps $f_i : (C, c) \to (X, x)$ which cover $C \times X$ (i.e., $\cup_i \mathrm{Graph}(f_i)$ is dense in $C \times X$).

We propose the following extension of Campana's conjecture.

Conjecture 2.2 (Inspired by Campana)**.** *The variety X is special if and only if it is geometrically-special.*

This conjecture is true in the basic cases discussed before.

Proposition 2.3 (De Franchis-Severi)**.** *If $\dim X = 1$, then X is geometrically special if and only if genus $(X) \leqslant 1$ if and only if X is special.*

Proposition 2.4. *If X is an abelian variety, then X is geometrically special.*

Proposition 2.5. *If X is rationally connected, then X is geometrically-special.*

The final product of our project (in progress) is the following result.

Theorem 2.6 (J.-Rousseau)**.** *If X is geometrically-special, then every linear quotient of $\pi_1(X)$ is virtually abelian.*

Our result is in accordance with our extension of Campana's conjecture (that a variety is special if and only if it is geometrically-special) and Campana's theorem that every linear quotient of the fundamental group of a special variety is virtually abelian.
The proof relies on several different ingredients. We mention a few in the next section.

3. Ingredients of the proof of Theorem 2.6

The structure of the proof of this theorem is analogous to the structure of the proof of Campana's (resp. Yamanoi's) theorems. All these proofs use properties of varieties which hold for special varieties, geometrically-special varieties, and varieties with a dense entire curve. That is, philosophically speaking, we exploit that geometrically-special varieties have many features in common with special varieties (but we do not use any actual properties of special varieties).

For example, building on work of Noguchi-Winkelmann-Yamanoi and Yamanoi, one can show that the Albanese map of a geometrically-special variety is surjective with connected fibres.

Theorem 3.1 (Albanese maps)**.** *Assume that X is special, or has a dense entire curve, or is geometrically-special. Then the Albanese map $X \to \mathrm{Alb}(X)$ is surjective with connected fibres, and has no multiple fibres in codimension one.*

When studying "virtual" properties of fundamental groups of geometrically-special varieties, it is natural to pass to finite etale covers. One may hope that by passing to such a covering the property of being geometrically-special is not lost. Indeed, we have the following result:

Theorem 3.2 (Inheriting "specialness")**.** *Let $Y \to X$ be a finite etale morphism. Then the following statements hold.*

(1) If X is special, then Y is special. (This is a theorem of Campana.)
(2) If X has a dense entire curve, then Y has a dense entire curve. (This is obvious.)
(3) If X is geometrically-special, then Y is geometrically-special.

There are two more ingredients. The following theorem follows from foundational result from Hodge theory proven by Deligne, Griffiths-Schmid, and Kang Zuo. It also uses the recent breakthrough by Bakker-Brunebarbe-Tsimerman on Griffiths's algebraicity conjectures for period maps.

Theorem 3.3 (Period maps)**.** *Assume that X is special, or has a dense entire curve, or is geometrically-special. Then every period map $X^{\mathrm{an}} \to \Gamma\backslash D$ is constant, i.e., every polarized $\mathbb{Z}$-variation of Hodge structures on X is constant.*

An important technical ingredient is the following result which we establish using Zuo's spectral covers, Kollár's Shafarevich maps, and Yamanoi's work on Nevanlinna theory.

Theorem 3.4 (p-Bounded representations)**.** *Let K be a finite extension of $\mathbb{Q}_p$ and let G be an almost simple algebraic gorup over K. If X is special, or has a dense entire curve, or is geometrically-special, then the image of every representation $\rho : \pi_1(X) \to G(K)$ with Zariski-dense image is contained in a maximal compact subgroup of $G(K)$. (One also says that ρ is p-bounded.)*

These are some of the ingredients necessary in our proof to show that every linear quotient of the fundamental group of a geometrically-special variety is virtually abelian. For details I refer to our upcoming paper.